PROBLEM 1. Recall that $\mathbb{N} = \{0, 1, 2, \dots\}$ denotes the set of nonnegative integers. Consider the following sets:

$$\begin{aligned}
A &= \{x \in \mathbb{Z} \mid x^2 \in \mathbb{N}\}, \\
B &= \{x \in \mathbb{N} \mid x \text{ is even}\} \cap \{x \in \mathbb{N} \mid x \text{ is a multiple of } 3\}, \\
C &= \{x \in \mathbb{N} \mid x \text{ is even}\} \cup \{x \in \mathbb{N} \mid x \text{ is a multiple of } 3\}, \\
D &= \{x \in \mathbb{N} \mid x \text{ is even}\} \triangle \{x \in \mathbb{N} \mid x \text{ is a multiple of } 3\}.
\end{aligned}$$

Write out some elements of each set and then describe the set in words, justifying your answer.

SOLUTION: We have $A = \{\dots, -2, -1, 0, 1, 2, \dots\} = \mathbb{Z}$. Indeed, the square of every integer is a nonnegative integer, so every $x \in \mathbb{Z}$ satisfies the condition $x^2 \in \mathbb{N}$.

We have

$$\begin{aligned}
B &= \{0, 2, 4, 6, 8, 10, 12, \dots\} \cap \{0, 3, 6, 9, 12, \dots\} \\
&= \{0, 6, 12, \dots\} \\
&= \{x \in \mathbb{N} \mid x \text{ is a multiple of } 6\}.
\end{aligned}$$

This is the case because an integer is a multiple of both 2 and 3 if and only if it is a multiple of $2 \cdot 3 = 6$.

We have

$$C = \{0, 2, 3, 4, 6, 8, 9, 10, 12, 14, 15, 16, \dots\}$$

These are the natural numbers divisible by 2 or 3 (or both).

We have

$$D = \{2, 3, 4, 8, 9, 10, 14, 15, \dots\}.$$

These are the natural numbers divisible by 2 or 3 but not by 6.

PROBLEM 2. Recall that De Morgan's law states that for all sets A, B, C,

$$C \setminus (A \cup B) = (C \setminus A) \cap (C \setminus B)$$

and

$$C \setminus (A \cap B) = (C \setminus A) \cup (C \setminus B).$$

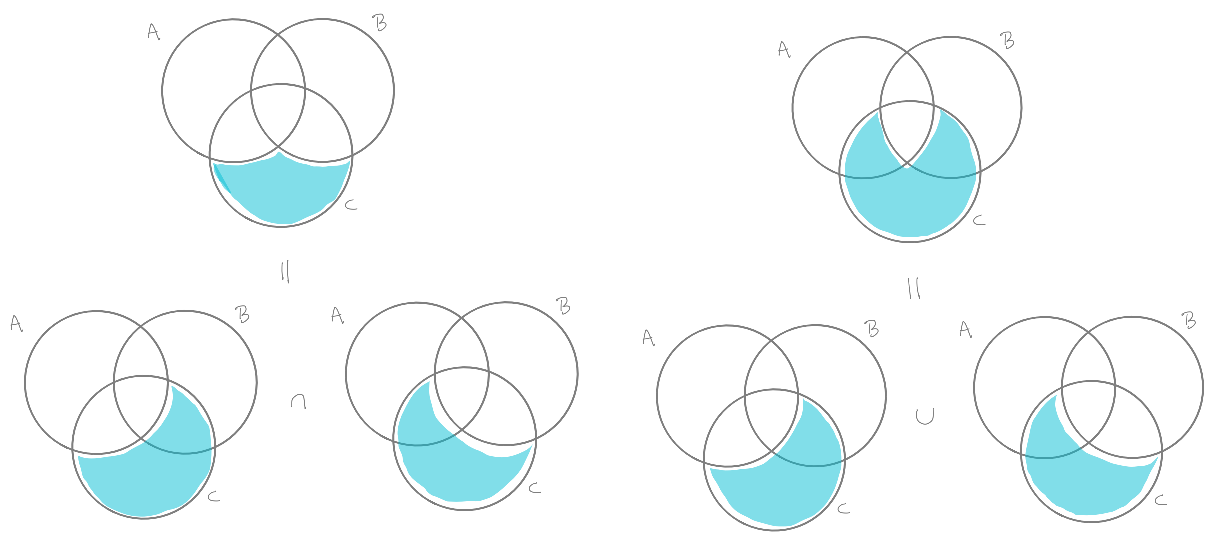
(a) Draw Venn diagrams that express these identities.

(b) Prove the first identity.

In order to prove an equality of sets $X = Y$, you can show $X \subseteq Y$ and $Y \subseteq X$.

SOLUTION:

(a) We offer the following Venn diagram cartoons illustrating De Morgan's law.



(b) ($\subseteq$) Suppose that x is a fixed but arbitrary element of $C \setminus (A \cap B)$. Then $x \in C$ and $x \notin A \cap B$. In order for x to not be an element $A \cap B$, it must not be an element of A or not be an element of B. Thus $x \in C \setminus A$ or $x \in C \setminus B$, *i.e.*, $x \in (C \setminus A) \cup (C \setminus B)$. We conclude that

$$C \setminus (A \cap B) \subseteq (C \setminus A) \cup (C \setminus B).$$

($\supseteq$) Suppose that x is a fixed but arbitrary element of $(C \setminus A) \cup (C \setminus B)$. Then $x \in C \setminus A$ or $x \in C \setminus B$. If $x \in C \setminus A$, then $x \in C$ and $x \notin A$; if instead $x \in C \setminus B$, we have that $x \in C$ and $x \notin B$. Note that in either case $x \in C$, and in addition we have that $x \notin A$ or $x \notin B$. This latter means that $x \notin A \cap B$ We conclude that $x \in C \setminus (A \cap B)$ and that

$$(C \setminus A) \cup (C \setminus B) \subseteq C \setminus (A \cap B).$$

Since we have proven both inclusions, we know that

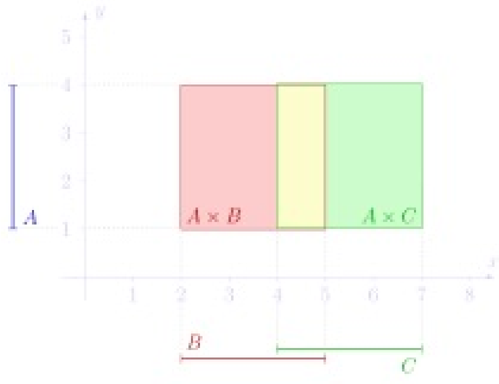
$$C \setminus (A \cap B) = (C \setminus A) \cup (C \setminus B).$$

PROBLEM 3. Suppose that A and B are finite sets with $|A| = m$, $|B| = n$, and $m \leq n$. What are the smallest and largest possible values of $|A \cap B|$?

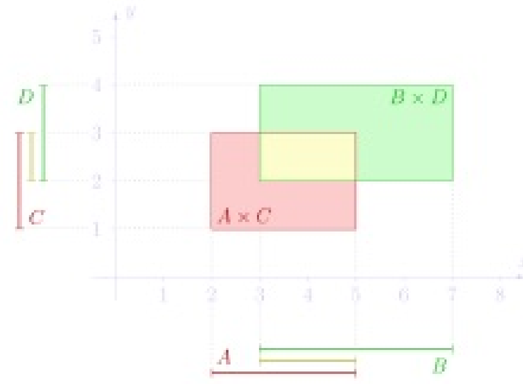
SOLUTION: If $A \cap B = \varnothing$, then $|A \cap B| = 0$, and this is the smallest possible value. If $A \subseteq B$, then $A \cap B = A$ and $|A \cap B| = m$; this is the largest possible value. We conclude that

$$0 \leq |A \cap B| \leq m.$$

PROBLEM 4. Explain how the following pictures illustrate the indicated identities, and then prove one or both of them.



$A \times (B \cap C) = (A \times B) \cap (A \times C)$



$(A \cap B) \times (C \cap D) = (A \times C) \cap (B \times D)$

SOLUTION: In the first picture, the red box represents $A \times B$ and the green box represents $A \times C$. The intersection of these boxes is the yellow region, and it agrees with $A \times (B \cap C)$.

In the second picture, the red box is $A \times C$ and the green box is $B \times D$. Their intersection is the yellow box, and this agrees with $(A \cap B) \times (C \cap D)$.

We now prove the first identity, again by demonstrating both inclusions. First suppose that (x, y) is a fixed but arbitrary element of $A \times (B \cap C)$. Then $x \in A$ and $y \in B \cap C$. Since $B \cap C \subseteq B$, we learn that $(x, y) \in A \times B$, and since $B \times C \subseteq C$, we learn that $(x, y) \in A \times C$. Since (x, y) is in both of these sets, it is also in their intersection. This shows that

$$A \times (B \cap C) \subseteq (A \times B) \cap (A \times C).$$

Now suppose that (x, y) is a fixed but arbitrary element of $(A \times B) \cap (A \times C)$. Then (x, y) is in both $A \times B$ and $A \times C$. Thus $x \in A$ and y is in both B and C. This precisely means that $(x, y) \in A \times (B \cap C)$, so

$$(A \times B) \cap (A \times C) \subseteq A \times (B \cap C).$$

Since we have demonstrated both inclusions, we learn that

$$A \times (B \cap C) = (A \times B) \cap (A \times C).$$

The proof of the second identity follows a similar pattern.