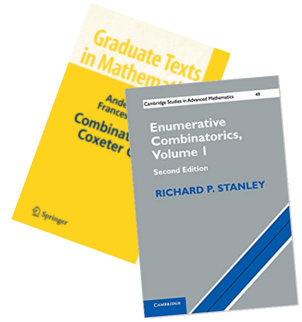
PROBLEM 1. You have nine math books. Five of them are yellow Springer-Verlag texts and four are gray Cambridge University Press texts.



(a) How many ways are there to arrange the books, left to right, along a shelf?

SOLUTION: There are nine choices for the first book. For each of these choices, there are eight choices for the second, and so on. The total number of ways of arranging the books is

$$9! := 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 362880.$$

(b) What if the yellow books need to stay together (but their ordering is still important)?

SOLUTION: We first treat the yellow books as one book so that we effectively have five books—one big yellow one and four gray ones. Reasoning as above, there are $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$ arrangements. We now choose the ordering for the yellow books. For each of the 5! arrangements, there are then 5! ways to order the five yellow books. By the multiplicative counting principle, the solution is thus $(5!)^2 = 14400$.

(c) What if, in addition, the gray books need to stay together (and ordering within each color group is important)?

SOLUTION: First consider the five yellow books as a single book and the four gray books as a single book. There are $2! = 2 \cdot 1$ to arrange these. For each of these arrangements, there are 5! ways in which to order the yellow books, and then, for each choice of an ordering of the yellow books, there are 4! ways to order the gray books. By the multiplicative counting principle, our solution is thus $2 \cdot 5! \cdot 4! = 5760$.

PROBLEM 2. A domino is a list of two, not necessarily distinct, numbers a, b where each of a and b are between 0 and 6, inclusive. We consider the pairs a, b and b, a to be the same.

(a) How many dominoes are there?

SOLUTION: We split this count into two cases: if $a = b$ and if $a \neq b$. There are seven dominos for which $a = b$, namely $0, 0$ through $6, 6$. If $a \neq b$, there are $7 \cdot 6 = 42$ choices for the *ordered* pair (a, b), and hence $42/2$ dominoes since each domino appears in 2 ordered pairs. The total number of dominoes is, thus, $7 + 21 = 28$.

(b) Say two dominoes *match* if they share at least one number. Thus, a matching pair will have the form

$$[a|b]\ [b|c]$$

where a, b, c are numbers between 0 and 6, inclusive. How many pairs of matching dominoes are there (where the order of the pair of dominoes does not count)? [Hints: A *double* is a domino with a repeated number, e.g., [4|4]. Why can't a matching pair consist of two doubles? Break the problem into two cases depending on whether a double occurs.]

SOLUTION: We will consider cases as suggested by the hint.

Case 1. Suppose no double occurs. Then a, b, c are three distinct numbers. There are seven choices for the central number, b. After this, we need to choose two numbers from six. (We can think of this as choosing a committee of two from six people.) There are six choices for a, and then five choices for c. Thus, there are a total of $6 \cdot 5 = 30$ choices. However, since we count $abbc$ as the same as $cbba$, we need to divide by 2. Thus, there 15 choices for the unordered pair a, c. Taking into account the seven choices for b, this first case gives

$$7 \cdot 15 = 105$$

matching pairs.

Case 2. In this case, one of the dominoes is a double, and the other isn't:

$$[a|a]\ [a|b].$$

There are seven choices for the double, i.e., for the choice of a. For each of these, there are six choices for b. That gives a total of

$$7 \cdot 6 = 42$$

matching pairs including a double.

Summing over the two cases gives the solution:

$$105 + 42 = 147.$$

Alternative solution: Here is a different solution that disregards the hint. Consider our pair of matching dominoes as pictured below:

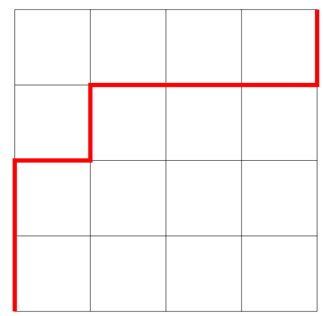
$$[a|b]$$

$$[b|c].$$

There are seven choices for b. Having chosen b, consider the numbers a and c—and here is the funny part—think of these two numbers as forming their own domino $[a|c]$. The only restriction on $[a|c]$ is that $a \neq c$ (since we need $[a|b] \neq [b|c]$). In other words $[a|c]$ is any non-double domino. Using the first part of this problem, we see there are 21 choices for $[a|c]$ for each of the seven choices for b. Therefore, the number of matching dominoes is

$$7 \cdot 21 = 147.$$

PROBLEM 3. A path on a square grid is called monotonic if it proceeds only by single steps right or up. On a 4×4 (or $n \times k$) grid, how many distinct monotonic paths go from the bottom left corner to the top right corner?



SOLUTION: We can consider a monotonic path as a sequence of four R's and four U's (right and up); in general n R's and k U's. For example, the sequence for the figure on the margin is

$$U\,U\,R\,U\,R\,R\,R\,U.$$

If we label the R's as R_1, R_2, R_3, R_4, and the U's as U_1, U_2, U_3, U_4, there are $8!$ ways of ordering these characters (just like in Problem 1). Since we want to treat the R's and the U's as being indistinguishable, we need to divide by our overcounting factor. Note that this corresponds to the number of ways of reordering the R's ($4!$) times the number of ways of reordering the U's ($4!$). Thus, the total number of paths is

$$\frac{8!}{4! \cdot 4!} = 70.$$

For general n and k the answer is

$$\frac{(n+k)!}{n! \cdot k!}.$$

Challenge

Challenge problems are optional and should only be attempted after completing the previous problems.

(a) In Problem 1, what if the only restriction is that the colors appear in a symmetrical pattern about the central book? [Hint: Let g stand for gray and y for yellow. Suppose the first four books have the color pattern $ggyy$. What is the rest of the pattern? How many arrangements have this color pattern? How many possible color patterns are there for the first four books?]

SOLUTION: The symmetry requirement says that the arrangement will have the form

$$a\ b\ c\ d\ e\ d\ c\ b\ a.$$

where each of a, b, c, d, e is either yellow or gray. Since there are an odd number of yellow books and an even number of gray books, the central book e must be yellow. Letting g stand for gray and y stand for yellow, the possible color patterns for $a\ b\ c\ d$ are

$$ggyy,\ gygy,\ gyyg,\ yggy,\ ygyg,\ yygg.$$

So there are 6 possibilities to consider.

Let's count the number of arrangements that start with the color pattern $ggyy$. The complete pattern of colors in that case is

$$g\ g\ y\ y\ y\ y\ y\ g\ g.$$

There are then $4! = 24$ ways to lay out the gray books, and for each of these there are 5! ways of laying out the yellow books. Therefore, the total number of choices is $4!\,5! = 24 \cdot 120 = 2880$.

The argument we just gave for the color pattern $ggyy$ applies equally well to the remaining five patterns, and so the count will be $4!\,5!$ in those cases, too. That means the total count is

$$6 \cdot 4!\,5! = 4!\,6! = 17280.$$