MATH 111: CALCULUS
HOMEWORK DUE FRIDAY WEEK 7

Make sure to review the homework instructions in the syllabus before writing your solutions. In particular, show your work and write in complete sentences (but also aim for concise explanations).

Problem 1. If a spherical tank of radius 4 feet has h feet of water to a height of h feet present in the tank, then the volume of the water in the tank is given by

$$V = \frac{\pi}{3} h^2 (12 - h)$$

in cubic feet. (You do not need to justify this, and may use it in your subsequent work.)

(a) What is the instantaneous rate of change of water in the tank with respect to height of the water at the instant $h = 1$ foot? Make sure you include units in your answer.

(b) Now suppose the height of the water is being regulated so that the height of the water at time t is given by

$$h(t) = \sin(\pi t) + 1$$

where t is measured in hours and $h(t)$ is still measured in feet. At what rate is the height of the water changing with respect to time at the instant $t = 2$? Include units.

(c) Continuing with the assumptions in (b), at what instantaneous rate is the volume of the water changing with respect to *time* at the instant $t = 2$?

(d) What are the main differences between the rates found in (a) and (c)? Include a discussion of the relevant units.

Problem 2. Let $f(x) = x + \sin x$.

(a) Use desmos or another graphing utility to sketch a graph of $y = f(x)$ and explain why f has an inverse function.

(b) Spend at least ten minutes attempting to produce an explicit formula for $f^{-1}(x)$ and then write "It appears that the inverse of f cannot be expressed in terms of the functions we usually encounter in this class."

(c) Let $g(x) = f^{-1}(x)$ and compute

$$g'\left(\frac{\pi}{4} + \frac{\sqrt{2}}{2}\right).$$

(d) What is the equation of the tangent line to $y = f^{-1}(x)$ at $x = \pi$? (Make sure you explain why your answer works despite some trouble with the formula for $g'(\pi)$.)

Problem 3. For a constant a, consider the curve D_a given by solutions to

$$y^2(y^2 - 1) = x^2(x^2 - a).$$

(a) Explain why $(0, 1)$ is a point on D_a for all a.

(b) Use implicit differentiation to determine the equation for the tangent line to D_a at $(0, 1)$.

(c) Use desmos or another graphing utility to sketch D_a and its tangent line at $(0, 1)$ for $a = 0, 1/2, 1, 2$.

Problem 4. Supoose $f\colon \mathbb{R} \to \mathbb{R}$ is differentiable and also equals its own inverse, that is,

$$f(f(x)) = x \quad \text{for all } x \text{ in } \mathbb{R}.$$

Note that such functions exist; for instance, $f(x) = x$ and $f(x) = -x$ are both differentiable with $f(f(x)) = x$.

(a) Find a third example of such a function. (Bonus: Find infinitely many such functions.)

(b) Explain why $f'(x) \neq 0$ for all x. (Hint: Apply the chain rule to $f(f(x)) = x$. Note: Your argument should work for *all* such f, not a specific example.)

(c) Call a real number a a *fixed point* of f when $f(a) = a$. Explain why

$$f'(a) = \pm 1$$

whenever a is a fixed point of f.