24.XII.4

Goals
- Exponential growth & decay
- Newton's law of cooling

Let $y(t)$ = size of population, varying over time

Suppose $y'(t) = k\,y(t)$ for some constant k, i.e.

"rate of growth is proportional to size of pop'n"

Then $y' = ky \Rightarrow \frac{y'}{y} = k \Rightarrow \int \frac{y'}{y}\,dt = \int k\,dt$

separable!

$u = y \quad du = y'\,dt$

$\Rightarrow \log(y) = kt + c$

$\Rightarrow \quad y = e^{kt+c}$ [exponentiate]

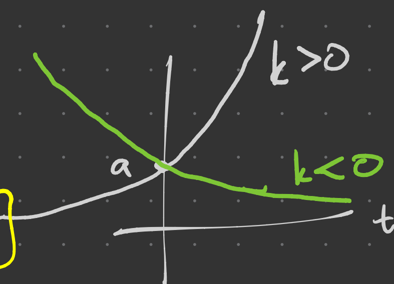
$= e^{c} e^{kt}$

$$\Longrightarrow \quad y = ae^{kt} \qquad [a = e^{c}]$$

Setting $t=0$, we see $y(0) = ae^{0} = a$, so a = initial pop'n.

<u>Q</u> If $y = ae^{kt}$, at what time t has the pop'n doubled? $(k>0)$

<u>A</u> Want t such that $y(t) = 2a$



$$ae^{kt} = 2a$$

$$e^{kt} = 2 \quad [a \neq 0]$$

$$kt = \log(2)$$

$$t = \frac{\log(2)}{k} \quad (\text{assuming } k \neq 0).$$

Newton's law of cooling

"carrying capacity"

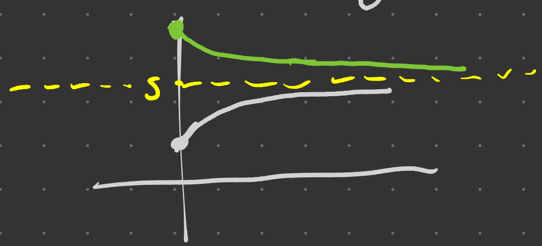
Now suppose $y' = r(S-y)$ for r, S positive constants.

Q When is the pop'n increasing? decreasing?

A y increasing $\iff y' > 0 \iff S - y > 0 \iff y < S$

y decreasing $\iff y > S$.

Q What is the long-term behavior of the pop'n?



Let's solve $y' = r(S-y)$:

$$\Rightarrow \frac{y'}{S-y} = r$$

separable!

$$\Rightarrow \int \frac{y'}{S-y}\,dt = \int r\,dt$$

$\{a\log(x) = \log(x^a)\}$

$RHS = rt + c$

$$LHS = -\int \frac{du}{u} = -\log(u) = -\log(S-y) \overset{!}{=} \log\left(\frac{1}{S-y}\right)$$

$u = S-y,\ du = -y'\,dt$

Thus $\log\left(\frac{1}{S-y}\right) = rt + c$

$$\Rightarrow \frac{1}{S-y} = e^{rt+c} = ae^{rt}$$ [exponentiate] $[a = e^c]$

$$\Rightarrow S - y = \frac{1}{ae^{rt}}$$

$$\Rightarrow y = S - \frac{1}{a}e^{-rt}$$

What is the relation b/w $I := y(0)$ and this?

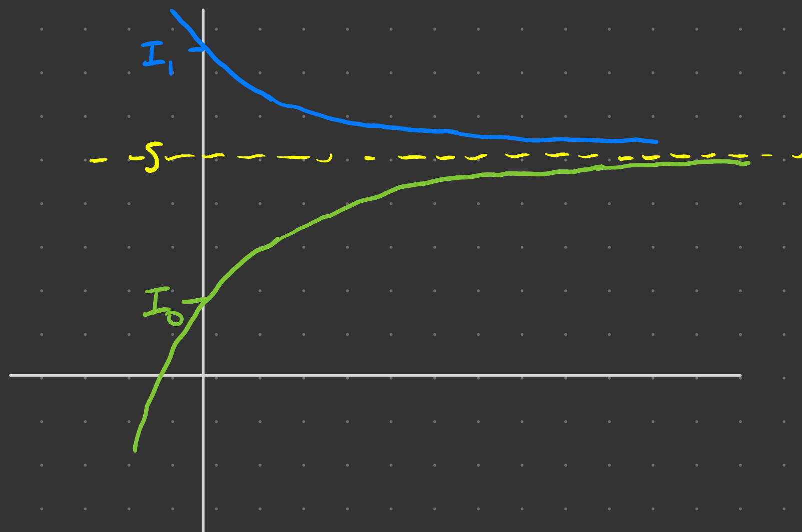
$$I = y(0) = S - \frac{1}{a} \Rightarrow a = \frac{1}{S-I}.$$

Thus $$\boxed{y = S + (I-S)e^{-rt}}$$

for I the initial pop'n.

Note $y \to S$ as $t \to \infty$. Call S the carrying capacity.

Rmk Could also think of S as room temperature, y the temperature of the liquid in a cup.



Q What does r control?

Problem You put a turkey in a 350°F oven initially at 40°F. After 10 minutes, the internal turkey temperature is 43°F. How long until the turkey is ready (165°F)?

$$43 = y(10) = 350 + (40 - 350)\, e^{-r \cdot 10}$$

use this to find r.