24. XII. 2

Goals
- differential equations & their solutions
- separable differentiable equations

Let y be a function of t. An equation in t and the derivatives of y is called a differential equation.

E.g. $y'' - y = 0$ (or, equivalently $y''(t) - y(t) = 0$)

$\iff y'' = y$.

What functions satisfy this? They are solutions to the differential equation.

For $y'' = y$, it turns out that all solutions are of the form

$$y = ae^{t} + be^{-t}$$

for some a, b constants.

Check that these work:

$$(ae^{t} + be^{-t})'' = (ae^{t} - be^{-t})'$$

$$= ae^{t} - (-be^{-t})$$

$$= ae^{t} + be^{-t} \checkmark$$

This is a "two-parameter family of solutions".

To specify a particular sol'n, we need two initial conditions.

For instance, $\underbrace{y(0)=0}_{\text{initial position}}$, $\underbrace{y'(0)=1}_{\text{initial velocity}}$.

$$0 = y(0) = ae^{0} + be^{-0} = a+b$$

$$y'(t) = ae^{t} - be^{-t}$$

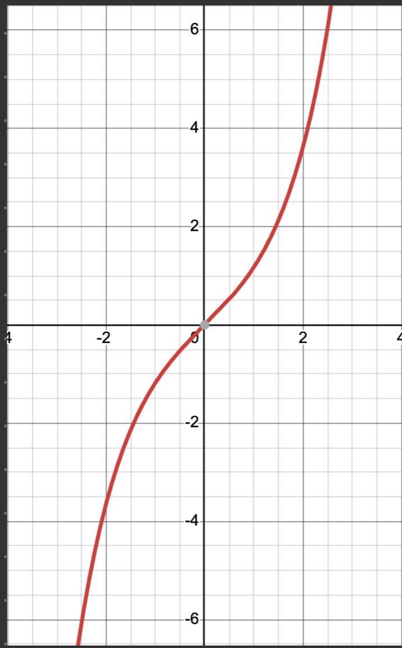
$$1 = y'(0) = a-b$$

$$2a = 1 \Rightarrow a = \frac{1}{2}$$

$$\Rightarrow b = \frac{-1}{2}$$

so for the given initial position and velocity, the unique sol'n is

$$y = \frac{1}{2}\left(e^{t} - e^{-t}\right)$$



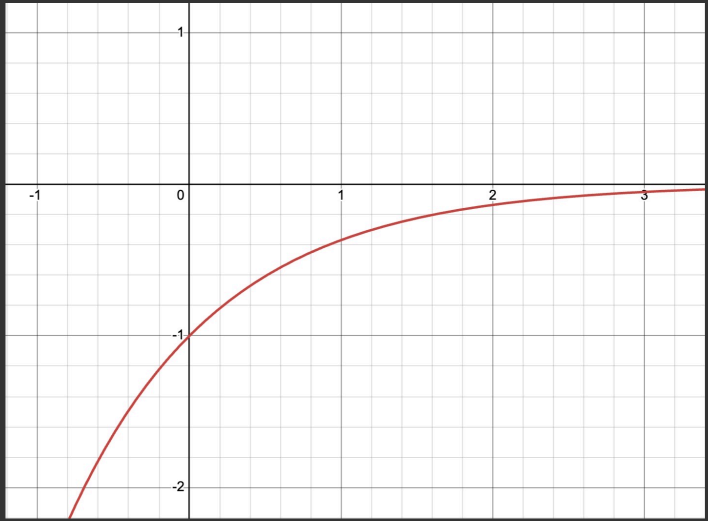
Q What if $y(0) = -1$, $y'(0) = 1$?

$$-1 = y(0) = a + b$$

$$+ \quad 1 = y'(0) = a - b$$

$$0 = 2a \Rightarrow a = 0 \Rightarrow b = -1.$$

A $y = -e^{-t}$



Q What if $y(0) = y'(0) = 0$?

A $a + b = y(0) = 0$

$a - b = y'(0) = 0$ $\Longrightarrow$ $a = b = 0 \Longrightarrow y = 0.$

E.g. First order equation $t y'(t) = 3y(t)$.

Then $\frac{y'}{y} = \frac{3}{t}$ —— the equation is separable: we can get y's on one side of the eq'n, t's on the other.

Integrate: $\int \frac{y'}{y}\, dt = \int \frac{3}{t}\, dt$

RHS $= 3\log(t) + C$

LHS $= \int \frac{1}{u}\, du = \log(u) + \tilde{C} = \log(y) + \tilde{C}$.

$u = y$, $du = y'\, dt$

Thus $\log(y) = 3\log(t) + k = \log(t^3) + k$

for some constant k.

Exponentiating,

$$e^{\log(y)} = e^{\log(t^3) + k}$$

$$\Rightarrow \quad y = e^{\log(t^3)} e^{k}$$

$$\Rightarrow \quad y = K t^3.$$ for K some positive constant

If $y(1) = 1$ then $K = 1$ and $y = t^3$.

Problem Solve $y' = \dfrac{3t}{y}$ (a) in general, and (b) subject to initial condition $y(0) = 5$.

$$yy' = 3t$$

$$\int yy'\,dt = \int 3t\,dt$$

$u = y$

$du = y'\,dt$

$$\int u\,du = 3\frac{t^2}{2} + c$$

$$\frac{u^2}{2} + \tilde{C} = 3\frac{t^2}{2} + c$$

$$\frac{y^2}{2} + \tilde{C} = \frac{3}{2}t^2 + c$$

$$y^2 = 3t^2 + k$$

$$y = \pm\sqrt{3t^2 + k}$$