24.XII.6

Goals · Logistic growth

Now consider the differential equation

$$y' = ry\underbrace{\left(1-\frac{y}{K}\right)}$$

for K some constant.

- if $y \ll K$ (y much less than K), this is ≈ 1

$\Rightarrow y' \approx ry$ — exponential

- but y close to K, this is ≈ 0

so growth slows near $y = K$.

To solve, separate:

$$\frac{y'}{y\left(1-\frac{y}{K}\right)} = r.$$

Partial fractions:

$$\frac{1}{y(1-y/K)} = \frac{A}{y} + \frac{B}{1-y/K}$$

(you're not responsible for this technique, but it's good to be aware of it!)

$$= \frac{A(1-y/K) + By}{y(1-y/K)}$$

So need $1 = A(1-y/K) + By = A + \left(-\frac{A}{K} + B\right)y$

$\iff A = 1, \; -\frac{A}{K} + B = 0$ b/c no y on LHS

$\iff A = 1, \; B = \frac{1}{K}$

So $$\frac{1}{y(1-y/K)} = \frac{1}{y} + \frac{1/K}{1-y/K}$$ [check!]

Back to diff'l eq'n:

$$\frac{y'}{y(1-\frac{y}{K})} = r \implies \int \frac{y'}{y(1-y/K)}\,dt = \int r\,dt = rt + c.$$

$$\text{LHS} = \int \left(\frac{y'}{y} + \frac{y'/K}{1-y/K} \right) dt$$

$$= \int \frac{y'}{y}dt + \frac{1}{K}\int \frac{y'}{1-y/K}\,dt$$

$u = 1-y/K \quad du = -y'/K\,dt \implies y'\,dt = -K\,du$

$$= \log(y) + \frac{1}{K}\int \frac{-K}{u}\,du = \log(y) - \frac{K}{K}\int \frac{1}{u}\,du$$

$$= \log(y) - \log(1-y/K) + \tilde{c}$$

$$= \log\left(y \cdot (1 - y/K)^{-1} \right) + \tilde{c}$$

Since LHS = RHS,

$$\log\left(y \cdot (1 - y/K)^{-1} \right) = rt + c$$

$$\Rightarrow \quad y(1 - y/K)^{-1} = e^{rt+c} = \underbrace{ae^{rt}}_{a = e^c}$$ [exponentiating]

$$\Rightarrow \quad \frac{Ky}{K-y} = ae^{rt}$$ [mult LHS by $\frac{K}{K}$]

$$\Rightarrow \quad Ky = ae^{rt}(K-y)$$

$$\Rightarrow \quad (K + ae^{rt})\, y = aKe^{rt}$$

$$\Rightarrow \quad y = \frac{aKe^{rt}}{ae^{rt} + K}$$

$$\Rightarrow \quad y = \frac{aK}{a + Ke^{-rt}}, \qquad \left[\text{mult by } \frac{e^{-rt}}{e^{-rt}}\right]$$

Let $I = y(0)$ and express a in terms of I:

$$I = y(0) = \frac{aK}{a+K}$$

$$\Rightarrow \quad I \cdot (a+K) = aK$$

$$\Rightarrow \quad IK = a(K-I)$$

$$\Rightarrow \quad a = \frac{IK}{K-I}.$$

Thus

$$\boxed{y = \frac{IK}{I + (K-I)e^{-rt}}}$$

<u>Note</u> As $t \to \infty$, $y \longrightarrow \dfrac{IK}{I+(K-I)\cdot 0} = K$ as expected.