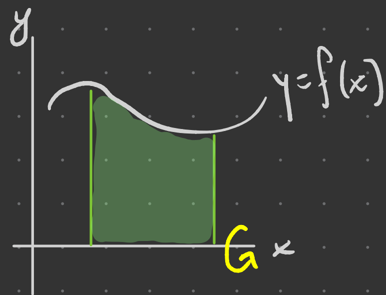
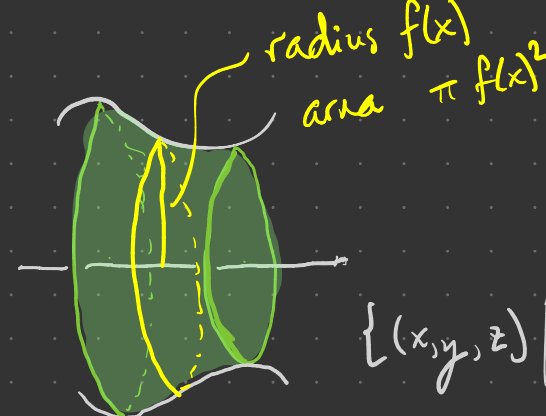
$$V = \int_{-1}^{1} \sqrt{3}\,(1-x^2)\,dx = \cdots$$

Solids of revolution

Take a region in the plane and rotate it about the x-axis:

E.g.



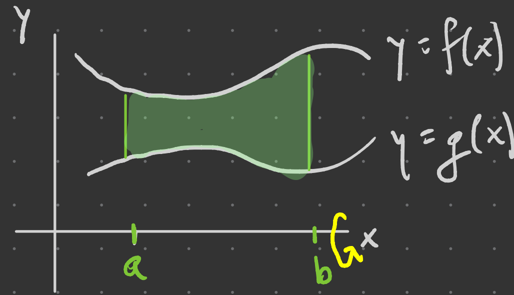


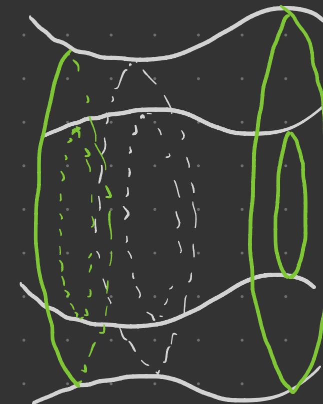
$$\left\{ (x,y,z) \;\middle|\; \begin{array}{l} a \le x \le b \\ y^2+z^2 \le f(x)^2 \end{array} \right\}$$

$$\boxed{V = \int_a^b \pi f(x)^2 \, dx}$$

"disk method"

E.g.





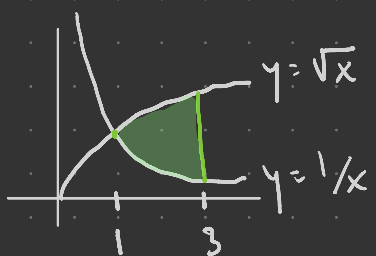
$$\boxed{V = \int_a^b \pi\left(f(x)^2 - g(x)^2\right)\,dx}$$

"washer method"

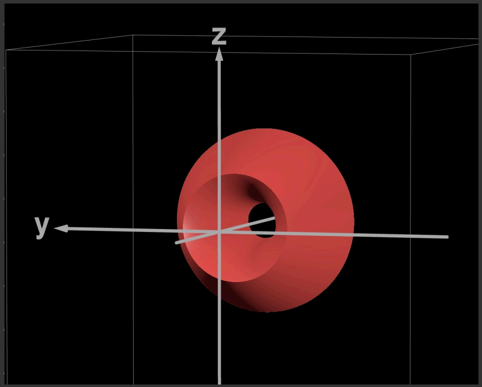
area

$= \pi\left(R^2 - r^2\right)$

E.g. Find the volume of a solid of revolution formed by revolving the region bounded by the graphs $y=\sqrt{x}$ and $y=\frac{1}{x}$ over the interval $[1,3]$ around the x-axis.



$\rightsquigarrow$



$$V = \int_1^3 \pi\left(\sqrt{x}^2 - \left(\frac{1}{x}\right)^2\right) dx = \pi\int_1^3 (x - x^{-2})\, dx$$

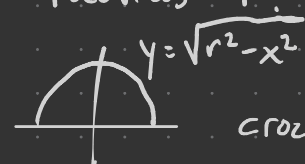
$$= \pi\left(\frac{1}{2}x^2 - \frac{x^{-1}}{-1}\right)\Bigg|_1^3 = \pi\left(\frac{1}{2}x^2 + \frac{1}{x}\right)\Bigg|_1^3$$

$$= \pi\left(\frac{9}{2} + \frac{1}{3} - \left(\frac{1}{2} + 1\right)\right)$$

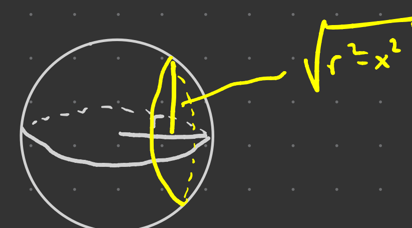
$$= \pi\left(3 + \frac{1}{3}\right)$$

$$\boxed{V = \frac{10\pi}{3}}$$

Problem Use the disk method to compute the volume of a ball of radius r.

Hint: rotate 

around x-axis.



$$V = \int_{-r}^{r} \pi \cdot \sqrt{r^2 - x^2}^{\,2}\, dx = \pi \int_{-r}^{r} (r^2 - x^2)\, dx = \pi\left(\left(r^2 x - \frac{x^3}{3}\right)\Big|_{-r}^{r}\right)$$

$$= \pi\left(r^3 - \frac{r^3}{3} - \left(-r^3 + \frac{r^3}{3}\right)\right)$$

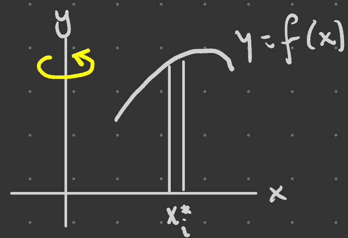
$$= \pi\left(2r^3 - \frac{2r^3}{3}\right)$$

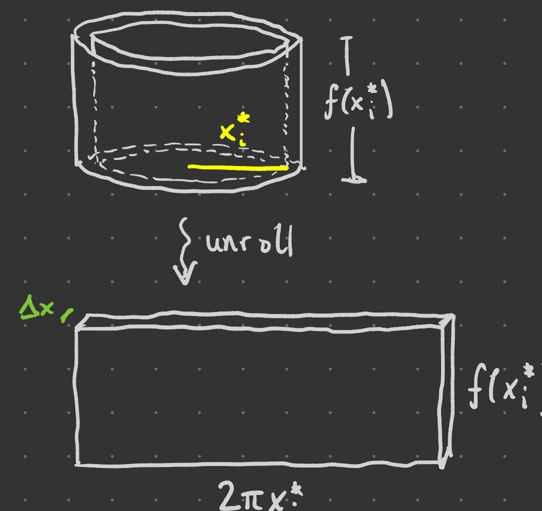
$$\boxed{V = \frac{4}{3}\pi r^3}$$

Cylindrical Shells

What if we rotate $\{(x,y) \mid a \le x \le b,\ 0 \le y \le f(x)\}$ around the y-axis?

Slice into "shells":





$$V_{shell} \approx 2\pi x_i^* f(x_i^*) \Delta x$$

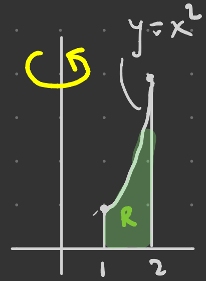
Thus $V \approx \sum_{i=1}^{n} 2\pi x_i^* f(x_i^*) \Delta x$

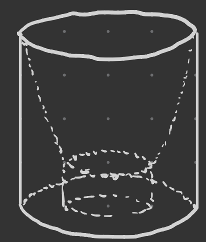
and $V = \lim_{n \to \infty} \sum_{i=1}^{n} 2\pi x_i^* f(x_i^*) \Delta x$

$$\Rightarrow \boxed{V = \int_a^b 2\pi x \cdot f(x)\, dx.}$$

"shell method"

E.g. Find the volume of the solid created by revolving $R = \{(x,y) \mid 1 \le x \le 2,\ 0 \le y \le x^2\}$ around the y-axis.

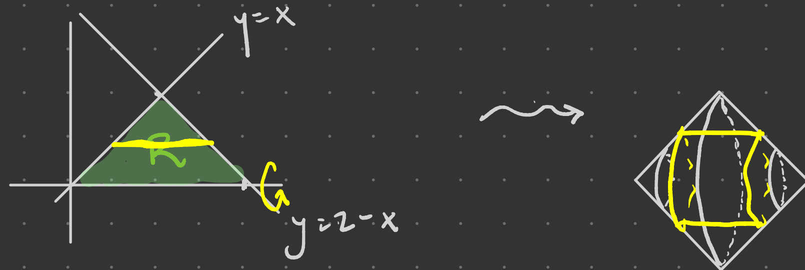




$$V = \int_1^2 2\pi x \cdot x^2\, dx = 2\pi \int_1^2 x^3\, dx = 2\pi \left.\frac{x^4}{4}\right|_1^2 = \frac{\pi}{2}(16-1)$$

$$= \boxed{\frac{15\pi}{2}}.$$

Problem Find the volume of the solid formed by revolving $R=\{(x,y) \mid 0\le x\le 2,\ 0\le y\le x, 2-x\}$ around the x-axis.



By disks:

$$V=\int_0^1 \pi x^2\,dx+\int_1^2 \pi(2-x)^2\,dx$$

By shells: $x=y$ $\quad x=2-y$

$$V=\int_0^1 2\pi y\,(2-y-y)\,dy$$

$$=\int_0^1 2\pi y\,(2-2y)\,dy$$