24. XI. 11

Goals • Integration by parts

If $h(x) = f(x)g(x)$, then $h'(x) = f'(x)g(x) + g'(x)f(x)$.

Thus $\int h'(x)\,dx = \int \left(g(x)f'(x) + f(x)g'(x)\right)dx$

i.e. $f(x)g(x) = \int g(x)f'(x)\,dx + \underbrace{\int f(x)g'(x)\,dx}_{\text{solve for this}}$

$$\Rightarrow \int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx.$$

Setting $u = f(x)$, $v = g(x)$

$du = f'(x)\,dx$, $dv = g'(x)\,dx$

this reads $\int u\,dv = uv - \int v\,du$.

Thm [Integration by parts] Let $u = f(x)$, $v = g(x)$ be functions with continuous derivatives. Then

$$\int u\,dv = uv - \int v\,du.$$

E.g. Let's compute $\int x \sin x\,dx$. Set $u = x$, $dv = \sin x\,dx$

$\Rightarrow du = dx$, $v = -\cos x$.

By integration by parts,

$$\underset{u}{\int x}\,\underbrace{\sin x\,dx}_{dv} = uv - \int v\,du$$

$$= -x\cos x + \int \cos x\,dx$$

$$= -x\cos x + \sin x + C.$$

Check $\quad (-x\cos x + \sin x)' = \cancel{-\cos x} - x(-\sin x) + \cancel{\cos x}$

$$= x\sin x$$

$$\int u\,dv = uv - \int v\,du$$

Problem Evaluate $\int x e^{2x}\,dx$ using $u = x$, $dv = e^{2x}\,dx$.

$$\Rightarrow du = dx, \quad v = \tfrac{1}{2}e^{2x}$$

$$\int x e^{2x}\,dx = \frac{1}{2} x e^{2x} - \int \frac{1}{2} e^{2x}\,dx$$

$$= \frac{1}{2} x e^{2x} - \frac{1}{4} e^{2x} + C$$

$$= \frac{1}{4} e^{2x}(2x-1) + C.$$

How do you choose u, dv? Try $u =$

L - logarithmic
I - inverse trig
A - algebraic
T - trig
E - exponential in that order.

E.g.

$$\int \frac{\log x}{x^3}\,dx = -\frac{1}{2}x^{-2}\log x + \int \frac{1}{2}x^{-3}\,dx$$

$u = \log x \quad dv = x^{-3}\,dx$

$du = \frac{1}{x}\,dx \quad v = \frac{x^{-2}}{-2}$

$$= \frac{-\log x}{2x^2} + \frac{1}{2}\,\frac{x^{-2}}{-2} + C$$

$$= \frac{-\log x}{2x^2} - \frac{1}{4x^2} + C$$

$$= \frac{2\log x - 1}{4x^2} + C$$

E.g. Evaluate $\int x^2 e^{3x}\,dx = \frac{1}{3}x^2e^{3x} - \frac{2}{3}\int xe^{3x}\,dx$.

$u = x^2$, $dv = e^{3x}\,dx$

$du = 2x\,dx$, $v = \frac{1}{3}e^{3x}$

Now $\int xe^{3x}\,dx = \frac{1}{3}xe^{3x} - \frac{1}{3}\int e^{3x}\,dx = \frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x} + C$

$u = x$ $dv = e^{3x}dx$

$du = dx$ $v = \frac{1}{3}e^{3x}$

so $\int x^2 e^{3x}\,dx = \frac{1}{3}x^2e^{3x} - \frac{2}{3}\left(\frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x}\right) + C$

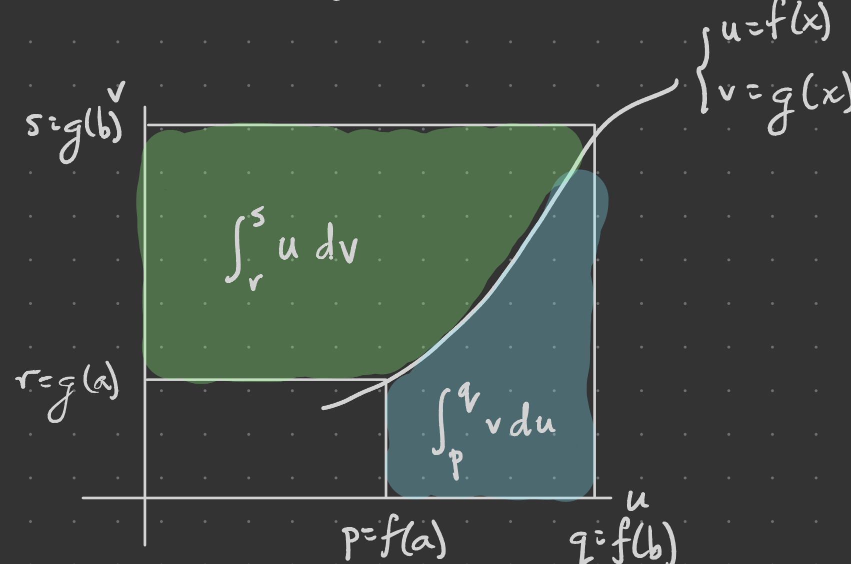
$= \frac{1}{3}x^2e^{3x} - \frac{2}{9}xe^{3x} + \frac{2}{27}e^{3x} + C.$

Thm [Integration by parts for definite integrals]

Let $u = f(x)$, $v = g(x)$ be functions with cts derivatives on $[a,b]$.

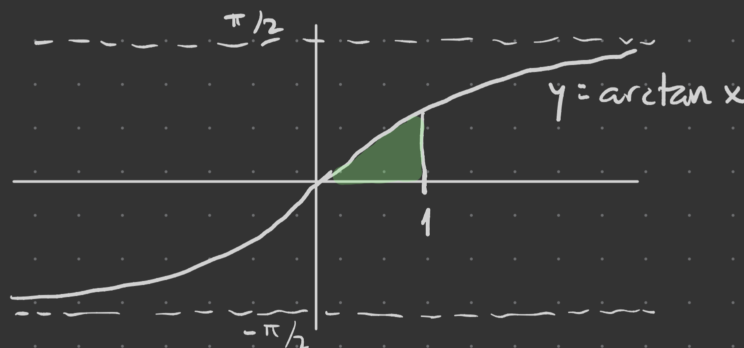
Then $$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du.$$

Pf by Picture



$$+ \quad = qs - pr = uv\Big|_a^b \qquad \square$$

E.g. Find the indicated area:



$$A = \int_0^1 \arctan x \, dx = x \arctan x \Big|_0^1 - \int_0^1 \frac{x}{1+x^2} \, dx$$

$u = \arctan x \qquad dv = dx$

$du = \frac{1}{1+x^2} \, dx \qquad v = x$

$u = 1 + x^2 \qquad du = 2x \, dx$

$$= \arctan(1) - \frac{1}{2}\int_1^2 \frac{du}{u}$$

$$= \frac{\pi}{4} - \frac{1}{2}\log u \Big|_1^2$$

$$= \frac{\pi}{4} - \frac{1}{2}\log(2) \approx 0.4388$$

Problem Determine $\displaystyle\int_0^{\pi/2} x \cos x \, dx$.