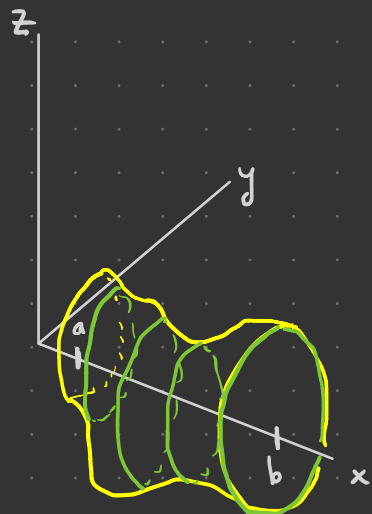
Goal

- Volume $= \int_a^b (\text{cross-sectional area})$
- Volume of solids of revolution by disks & washers



area of cross-section with $x = x_i^*$

volume $V \approx \sum_{i=1}^{n} A(x_i^*)\,\Delta x$

and $V = \lim_{n\to\infty} \sum_{i=1}^{n} A(x_i^*)\,\Delta x$

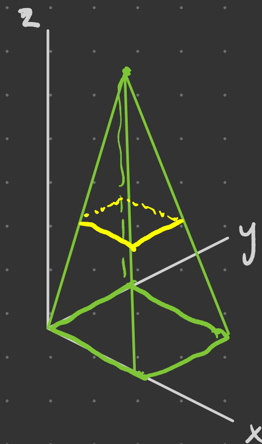
$= \int_a^b A(x)\,dx$

Slicing Method

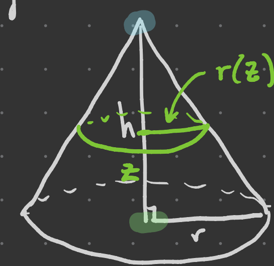
① Examine the solid and determine the shape of cross-sections (with respect to some coordinate axis, probably).

② Determine a formula for area of cross-section

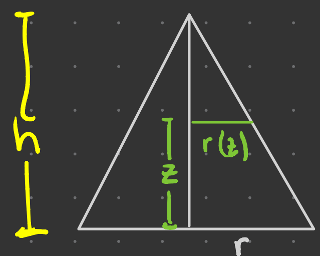
③ Integrate the area formula over the appropriate interval to get volume.



E.g. Let's find the volume of a circular cone with base radius r, height h:



$A(z) = \pi r(z)^2$ — but what is $r(z)$?



By similar triangles, $\dfrac{h-z}{r(z)} = \dfrac{h}{r}$

$\Rightarrow r(z) = \dfrac{r}{h}(h-z) = r - \dfrac{r}{h}z$

Thus $A(z) = \pi\left(r - \frac{r}{h}z\right)^2$ and $V = \displaystyle\int_0^h A(z)\,dz = \int_0^h \pi\left(r - \frac{r}{h}z\right)^2 dz$

$\underline{u = r - \frac{r}{h}z}$

$du = -\frac{r}{h}\,dz$

bounds describe smallest & largest z-values when we slice thru shape

$\Rightarrow dz = \frac{-h}{r}\,du$

So by substitution, $V = \pi \int_{r = u(0)}^{0 = u(h)} u^2 \cdot \left(\frac{-h}{r}\right) du$

$$= -\frac{\pi h}{r} \frac{u^3}{3} \Big|_r^0$$

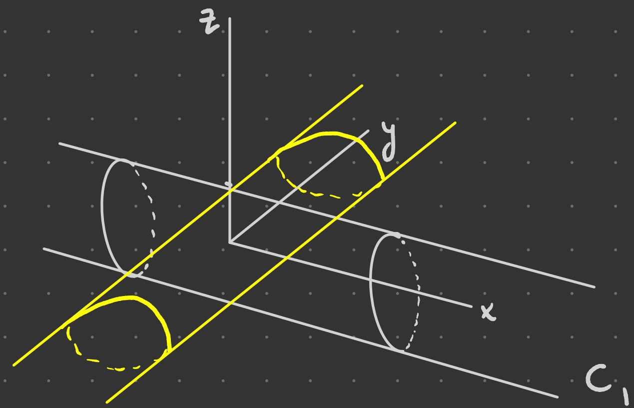
$$= \frac{\pi h}{r} \frac{r^3}{3}$$

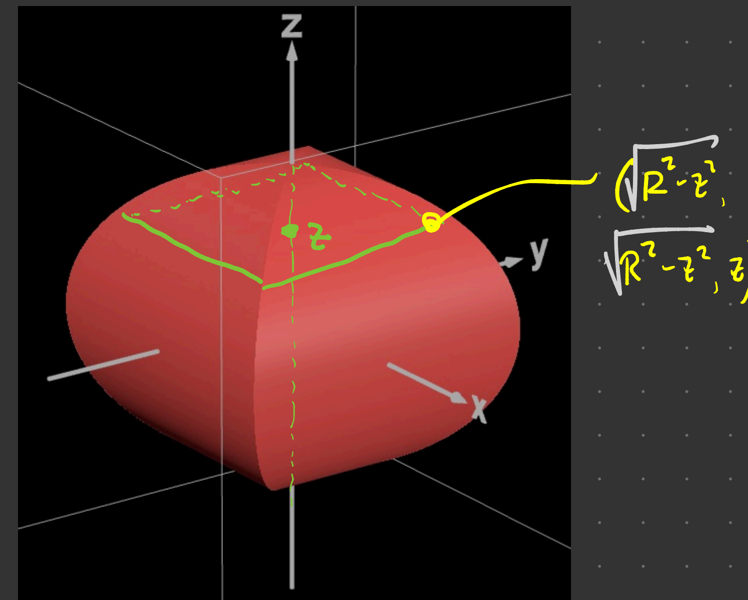
$$\boxed{V = \frac{1}{3}\pi r^2 h}$$

E.g. Consider cylinders $C_1 = \{(x,y,z) \mid y^2 + z^2 \leq R^2\}$

$C_2 = \{(x,y,z) \mid x^2 + z^2 \leq R^2\}$.

What is the volume of $C_1 \cap C_2 = \{(x,y,z)$ in both C_1 and $C_2\}$?





Q Along which axis should we slice?

A Perpendicular to z-axis so that slices are squares!

side length at height z is $2\sqrt{(R^2-z^2)}$ so $A(z) = 4(R^2-z^2)$

Thus $V = \int_{-R}^{R} 4\left(R^2 - z^2\right) dz = 4\int_{-R}^{R} R^2 \, dz - 4\int_{-R}^{R} z^2 \, dz$

$$\cancel{= 4\int_{-R}^{R} \left(R^4 - 2R^2 z^2 + z^4\right) dz}$$

$$\cancel{= 4\left(R^4 z - \frac{2}{3} R^2 z^3 + \frac{1}{5} z^5\right)\Big|_{-R}^{R}}$$

$$\cancel{= 8\left(R^5 - \frac{2}{3} R^5 + \frac{1}{5} R^5\right)}$$

$$= 4R^2 z\Big|_{-R}^{R} - 4\frac{z^3}{3}\Big|_{-R}^{R}$$

$$= 8R^3 - \frac{8}{3} R^3$$

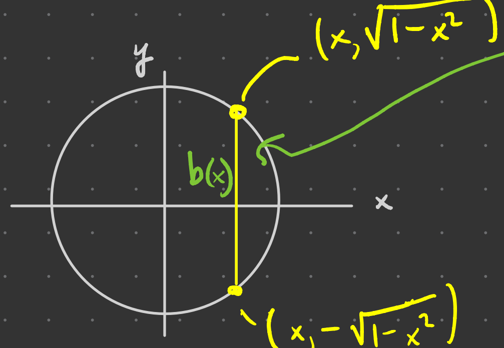
$$= \frac{16}{3} R^3$$

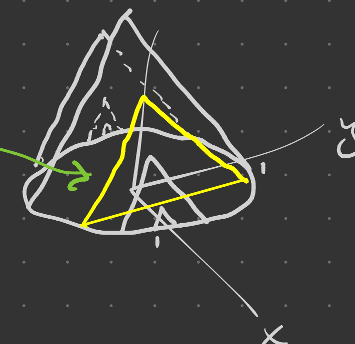
$$\boxed{V = \frac{16}{3} R^3}$$

(no π!)

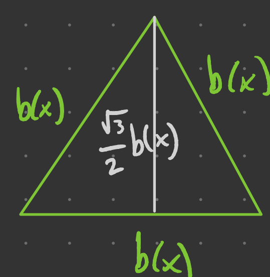
Problem Find the volume of the solid with base the disk of radius 1 in the xy-plane, cross-sections perpendicular to x-axis equilateral triangles.

From above:





$$b(x) = 2\sqrt{1-x^2}$$



$$A(x) = \frac{1}{2} b h$$

$$= \frac{1}{2} \frac{\sqrt{3}}{2} b(x) \cdot b(x)$$

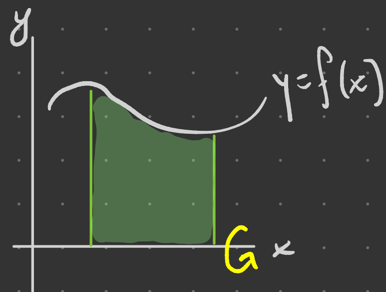
$$= \frac{\sqrt{3}}{4} b(x)^2 = \sqrt{3}\,(1-x^2)$$

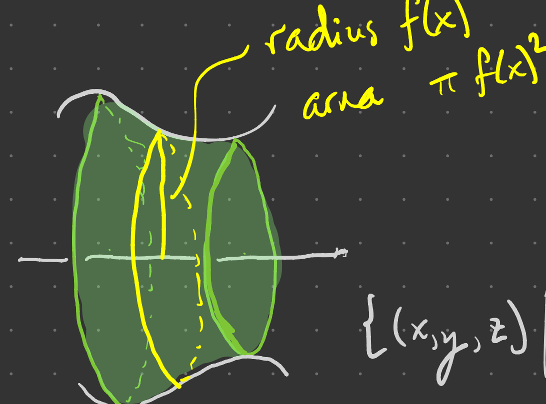
Solids of revolution

$$V = \int_{-1}^{1} \sqrt{3}\,(1-x^2)\,dx = \cdots$$

Take a region in the plane and rotate it about the x-axis:

E.g.



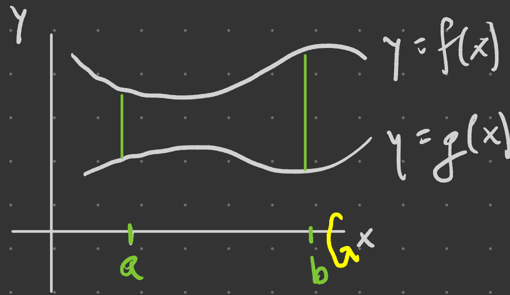


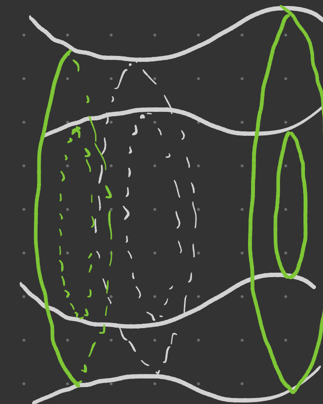
$$\{(x,y,z) \mid a\le x\le b,\ y^2+z^2\le f(x)^2\}$$

$$\boxed{V=\int_a^b \pi f(x)^2\,dx}$$

"disk method"

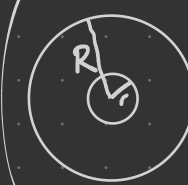
E.g.





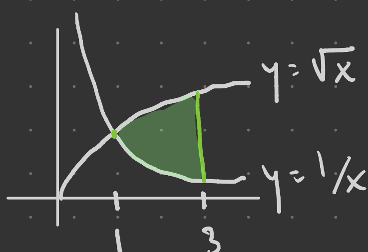
$$\boxed{V = \int_a^b \pi\left(f(x)^2 - g(x)^2\right) dx}$$

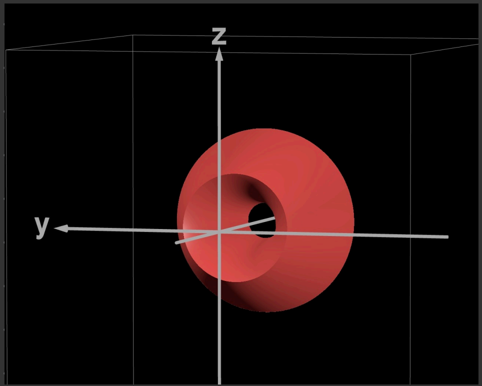
"washer method"

area ()

$= \pi\left(R^2 - r^2\right)$

E.g. Find the volume of a solid of revolution formed by revolving the region bounded by the graphs $y=\sqrt{x}$ and $y=\frac{1}{x}$ over the interval $[1,3]$ around the x-axis.





$$V=\int_1^3 \pi\left(\sqrt{x}^2-\left(\frac{1}{x}\right)^2\right)dx = \pi\int_1^3 (x-x^{-2})\,dx$$

$$= \pi\left(\frac{1}{2}x^2 - \frac{x^{-1}}{-1}\right)\Bigg|_1^3 = \pi\left(\frac{1}{2}x^2 + \frac{1}{x}\right)\Bigg|_1^3$$

$$= \pi\left(\frac{9}{2} + \frac{1}{3} - \left(\frac{1}{2} + 1\right)\right)$$

$$= \pi\left(3 + \frac{1}{3}\right)$$

$$\boxed{V = \frac{10\pi}{3}}$$

Problem Use the disk method to compute the volume of a ball of radius r.