24. XI. 8

Goals
- Evaluate indefinite and definite integrals via substitution

Recall $\underbrace{\int f(x)\,dx}_{\text{indefinite integral of } f} = F(x) + C$ means that $F'(x) = f(x)$.

Differention rules $\rightsquigarrow$ **Antidifferentiation rules**

$$\frac{d}{dx} x^n = n x^{n-1} \qquad \int x^n\,dx = \frac{x^{n+1}}{n+1} + C$$

linearity

$$\frac{d}{dx}(f+g) = \frac{df}{dx} + \frac{dg}{dx} \qquad \int (f(x)+g(x))\,dx = \int f(x)\,dx + \int g(x)\,dx$$

$$\frac{d}{dx}[cf] = c\frac{df}{dx} \qquad \int cf(x)\,dx = c\int f(x)\,dx$$

chain rule:

$$(f\circ g)'(x) = f'(g(x))\, g'(x)$$

substitution

??

product rule

$$(f\cdot g)' = f'\cdot g + f\cdot g'$$

integration by parts

??

Thm [Substitution for indefinite integrals] Suppose f, g' cts.

If $F' = f$, then $\int f(g(x))\cdot g'(x)\,dx = F(g(x)) + C$.

By setting $u = g(x)$, $du = g'(x)\,dx$, we can also write this as

$$\int f(g(x))\, g'(x)\,dx = \int f(u)\,du.$$

Pf By the chain rule,

$$\frac{d}{dx} F(g(x)) = F'(g(x))\, g'(x)$$
$$= f(g(x))\, g'(x).$$

Integrating,

$$F(g(x)) + C = \int f(g(x))\, g'(x)\, dx. \quad \square$$

E.g.

$$\int \underbrace{(x^2-3)^3}_{u} \cdot \underbrace{2x\, dx}_{du} = \int u^3\, du$$
$$= \frac{u^4}{4} + C$$
$$= \frac{(x^2-3)^4}{4} + C.$$

$$\int f(\underbrace{g(x)}_{u})\, \underbrace{g'(x)dx}_{du}$$
$$= F(g(x)) + C$$
$$= F(u) + C$$

for $F' = f$

Substitution Strategy

(1) Look for $g(x)$ in integrand such that $\approx g'(x)$ is also in the integrand.

(2) Subtitute $u = g(x)$, $du = g'(x)\,dx$ into the integral

(3) Integrate with respect to variable u.

(4) Rewrite the result in terms of x

E.g.

$$\int 3x^2 (x^3-3)^2\,dx = \int u^2\,du$$

$u = x^3 - 3$

$du = 3x^2\,dx$

$$= \frac{u^3}{3} + C$$

$$= \frac{(x^3-3)^3}{3} + C$$

Check

$$\frac{d}{dx}\frac{(x^3-3)^3}{3}$$

$$= \frac{\cancel{3}(x^3-3)^2}{\cancel{3}} \cdot 3x^2$$

$$= (x^3-3)^2 \cdot 3x^2 \quad \checkmark$$

E.g. $\int z\sqrt{z^2-5}\,dz = ?$

Set $u = z^2-5$, $du = 2z\,dz \Longrightarrow z\,dz = \frac{1}{2}du$.

Thus $\int z\sqrt{z^2-5}\,dz = \int \frac{1}{2}u^{1/2}\,du$

$$= \frac{1}{2}\frac{u^{3/2}}{3/2} + C$$

$$= \frac{1}{3}u^{3/2} + C$$

$$= \frac{1}{3}(z^2-5)^{3/2} + C.$$

Problems (1) Find $\int x^2(x^3+5)^9\,dx$

(2) Find $\int \frac{\cos t}{\sin^2 t}\,dt$

(1) Set $u = x^3+5$, so $du = 3x^2\,dx \Rightarrow \frac{1}{3}du = x^2 dx$.

By substitution,

$$\int x^2(x^3+5)^9\,dx = \int \frac{1}{3}u^9\,du$$

$$= \frac{1}{3}\,\frac{u^{10}}{10} + C$$

$$= \frac{1}{30}(x^3+5)^{10} + C$$

(2) $$\int \frac{\cos t}{\sin^2 t}\,dt = \int \cos t \cdot \frac{1}{\sin^2 t}\,dt = \int \frac{1}{u^2}\,du = \int u^{-2}\,du$$

$u = \sin t$

$du = \cos t\,dt$

$$= \frac{u^{-1}}{-1} + C = \frac{-1}{\sin t} + C$$

E.g.

$$\int \frac{x}{\sqrt{x-1}}\,dx = \int \frac{u+1}{\sqrt{u}}\,du = \int \frac{u+1}{u^{1/2}}\,du$$

$u = x-1 \Rightarrow x = u+1$

$du = dx$

$= -\csc t + C$

$$\frac{u+1}{\sqrt{u}} = \frac{u}{\sqrt{u}} + \frac{1}{\sqrt{u}}$$

$$= \frac{u}{u^{1/2}} + \frac{1}{u^{1/2}}$$

$$= u^{1/2} + u^{-1/2}$$

$$= \int \left(u^{1/2} + u^{-1/2}\right) du = \frac{u^{3/2}}{3/2} + \frac{u^{1/2}}{1/2} + C$$

$$= \frac{2}{3}u^{3/2} + 2u^{1/2} + C$$

$$= \frac{2}{3}(x-1)^{3/2} + 2(x-1)^{1/2} + C$$

$$= \frac{2}{3}(x-1)^{1/2}\left((x-1)+3\right) + C$$

$$= \frac{2}{3}(x-1)^{1/2}(x+2) + C.$$

Problem Check this answer!

$$\frac{d}{dx}\left[\frac{2}{3}(x-1)^{3/2}+2(x-1)^{1/2}\right]=(x-1)^{1/2}+(x-1)^{-1/2}$$

$$=\sqrt{x-1}+\frac{1}{\sqrt{x-1}}=\frac{x-1+1}{\sqrt{x-1}}$$

$$=\frac{x}{\sqrt{x-1}} \quad \checkmark$$

Thm [Substitution for definite integrals]

Let $u=g(x)$ with g' continuous on $[a,b]$. Let f be continuous over the image of $u=g(x)$. Then

$$\int_a^b f(g(x))\,g'(x)\,dx=\int_{g(a)}^{g(b)} f(u)\,du.$$

Pf If $\int f(u)\,du = F(u) + C$, then

$$\int_a^b f(g(x))\,g'(x)\,dx = F(g(x))\Big|_{x=a}^{x=b}$$

$$= F(g(b)) - F(g(a))$$

$$= \int_{g(a)}^{g(b)} f(u)\,du. \quad \square$$

E.g.

$$\int_{-1}^{0} y(2y^2-3)^5\,dy = \int_{2(-1)^2-3}^{2(0)^2-3} \frac{1}{4}u^5\,du = \int_{-1}^{-3} \frac{1}{4}u^5\,du$$

$2(0)^2-3 = u(0)$

$2(-1)^2-3 = u(-1)$

$u = 2y^2-3$

$du = 4y\,dy$

$y\,dy = \frac{1}{4}\,du$

$$= \frac{1}{4} \frac{u^6}{6} \Big|_{u=-1}^{u=-3}$$

$$= \frac{1}{24} \left((-3)^6 - (-1)^6 \right)$$

$$= \frac{91}{3}.$$

Problem Evaluate $\int_0^1 x^2 \cos\left(\frac{\pi}{2} x^3\right) dx$.

E.g. Find $\int_0^{\pi/2} \cos^2\theta \, d\theta$.

First, $\cos^2\theta = \dfrac{1+\cos 2\theta}{2}$, so

$$\int_0^{\pi/2} \cos^2\theta \, d\theta = \int_0^{\pi/2} \left(\frac{1}{2} + \frac{1}{2}\cos 2\theta\right) d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} d\theta + \frac{1}{2}\int_0^{\pi/2} \cos 2\theta \, d\theta$$

$$= \frac{\pi}{4} + \frac{1}{2} I .$$

Now using $u = 2\theta$, $du = 2\,d\theta$,

$$I = \int_0^{\pi} \frac{1}{2}\cos u \, du = \frac{1}{2}\sin u \Big|_{u=0}^{u=\pi}$$

$$= \frac{1}{2}(\sin\pi - \sin 0)$$

$$= 0.$$

Thus $\displaystyle\int_0^{\pi/2} \cos^2\theta \, d\theta = \frac{\pi}{4} + \frac{1}{2} I$

$$= \frac{\pi}{4}.$$

$$\log(x) := \int_1^x \frac{1}{t}\,dt$$

$$\log(ab) = \int_1^{ab} \frac{1}{t}\,dt = \int_1^{a} \frac{1}{t}\,dt + \int_a^{ab} \frac{1}{t}\,dt$$

$$u = \frac{1}{a}t \qquad du = \frac{1}{a}\,dt$$

$$\Rightarrow t = au \qquad dt = a\,du$$

$$= \log(a) + \int_1^{b} \frac{1}{au}\,a\,du$$

$$= \log(a) + \log(b)$$