24. X. 30

Goals
- Define integrability and integrals
- Definite integrals and signed area and averages

Recall that for $f:[a,b) \to \mathbb{R}$ a function, a Riemann sum for f (relative to a regular partition $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ and choices $x_i^* \in [x_i, x_{i-1}]$, $1 \le i \le n$) is

$$\sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

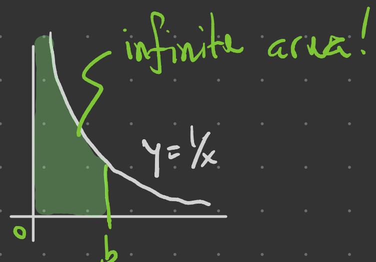
where $\Delta x = x_i - x_{i-1} = \dfrac{b-a}{n}$.

Defn If $\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$ exists and is independent of choice of x_i^*, then we say that f is **integrable** (over $[a,b]$) and write

$$\boxed{\int_a^b f(x)\,dx := \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x}$$

⚠2 Not all functions are integrable! E.g. $f(x) = \begin{cases} 1/x & x>0 \\ 0 & x=0 \end{cases}$

is not integrable over $[0,b]$ for any $b>0$.



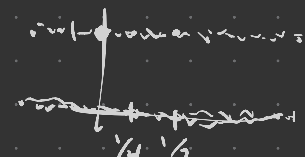
⚠⚠2 But $\frac{1}{\sqrt{x}}$ $\underline{\underline{\text{is}}}$ integrable over $[0,b]$ despite "blowing up."

⚠⚠⚠2 Some functions have different limits of Riemann sums depending on choices of x_i^*!

E.g. the function $\chi_{\mathbb{Q}} : \mathbb{R} \to \mathbb{R}$ defined by

characteristic function of the rational numbers $\mathbb{Q}$

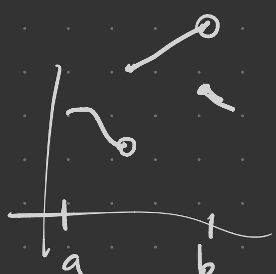
$$\chi_{\mathbb{Q}}(x) = \begin{cases} 0 & \text{if } x \text{ is irrational} \\ 1 & \text{if } x \text{ is rational.} \end{cases}$$



Over $[0,1]$, if we take x_i^* rational all Riemann sums evaluate to 1; if all x_i^* irrational, they're all 0. 😱

<u>Fact</u> If $f : [a,b] \to \mathbb{R}$ is continuous (or piecewise continuous), then f is integrable.

prove in Math 112.



Terminology

$\sum \underset{n\to\infty}{\rightsquigarrow} \int$: stylized summa

$$\underbrace{\overset{\sum}{\int_a^b}}_{\text{limits of integration}} \underbrace{\overset{f(x_i^*)}{f(x)}}_{\text{integrand}} \; \underbrace{\overset{\Delta x}{dx}}_{\text{variable of integration}}$$

(definite) integral of f from a to b

Note $\displaystyle \int_a^b f(x)\,dx = \int_a^b f(u)\,du = \int_a^b f(\text{🧯})\,d\text{🧯}$

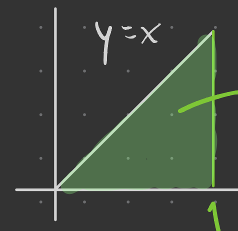
E.g. Monday we saw that via right Riemann sum

$x_i^* = \frac{i}{n}$, for $f(x) = x$,

$$\sum_{i=1}^{n} f(x_i^*)\Delta x = \sum_{i=1}^{n} \frac{i}{n}\cdot\frac{1}{n} = \frac{n-1}{2n} \xrightarrow[n\to\infty]{} \frac{1}{2}.$$

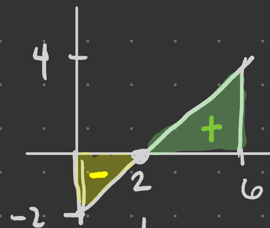
Thus $\displaystyle\int_0^1 x\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x = \lim_{n\to\infty} \frac{n-1}{2n} = \frac{1}{2}.$

Does this make sense?



area $\frac{1}{2}\cdot 1\cdot 1 = \frac{1}{2}$ ✓

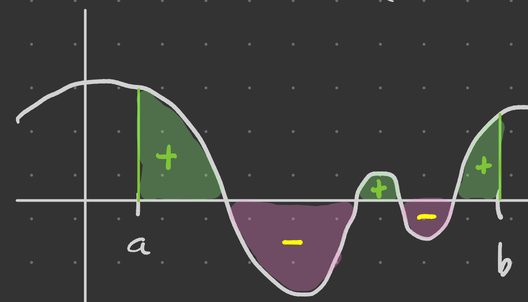
Problem Use geometry to find $\displaystyle\int_0^6 (x-2)\,dx$.



$$\int_0^6 (x-2)\,dx = \frac{1}{2}\cdot 4\cdot 4 - \frac{1}{2}\cdot 2\cdot 2 = 8-2 = 6$$

Note

$$\int_a^b f(x)\,dx = \left(\begin{array}{l}\text{area above x-axis}\\ \text{and below } y=f(x)\end{array}\right) - \left(\begin{array}{l}\text{area below x-axis}\\ \text{and above } y=f(x)\end{array}\right)$$

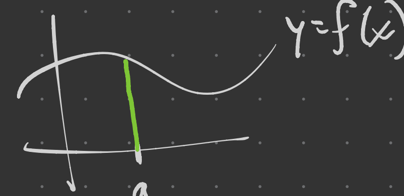


area b/w $y=f(x)$
and x-axis

$$= \int_a^b |f(x)|\,dx$$

Properties of $\int_a^b$:

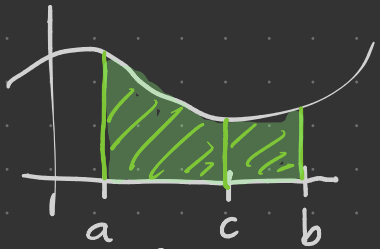
- $\int_a^a f(x)\,dx = 0$



- $\displaystyle\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$ [convention]

- $\displaystyle\int_a^b (f(x) + c\,g(x))\,dx = \int_a^b f(x)\,dx + c\int_a^b g(x)\,dx$ [linearity]

- $\displaystyle\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$



__Problem__ Suppose $\displaystyle\int_1^5 f(x)\,dx = -3$, $\displaystyle\int_2^5 f(x)\,dx = 4$. Find $\displaystyle\int_1^2 f(x)\,dx$

__A__

$$\int_1^5 f(x)\,dx = \int_1^2 f(x)\,dx + \int_2^5 f(x)\,dx$$

$$-3 = \int_1^2 f(x)\,dx + 4 \implies \int_1^2 f(x)\,dx = -3-4 = -7$$

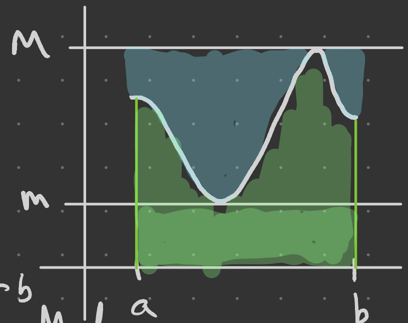
Comparison Theorem If $f(x) \geq g(x)$ for all $a \leq x \leq b$, then

$$\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx.$$

Cor (1) If $f(x) \geq 0$ on $[a,b]$, then $\int_a^b f(x)\,dx \geq 0$.

(2) If $m \leq f(x) \leq M$ on $[a,b]$, then

$$m\cdot(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a).$$

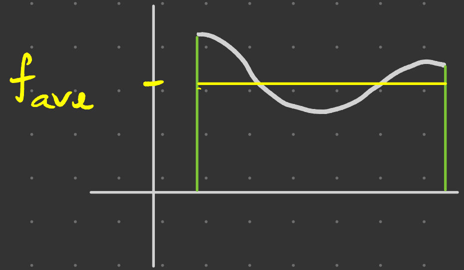


$$\int_a^b m\,dx = m\cdot(b-a) \leq f(x) \leq M\cdot(b-a) = \int_a^b M\,dx$$

Defn The average value of $f(x)$ on $[a,b]$ is

$$f_{ave} := \frac{1}{b-a}\int_a^b f(x)\,dx.$$

Note f_{ave} = (height of rectangle whose signed area = signed area b/w $y=f(x)$ and x-axis)



Problem Show

$$\int_a^b \left(f(x) - f_{ave}\right) dx = 0.$$

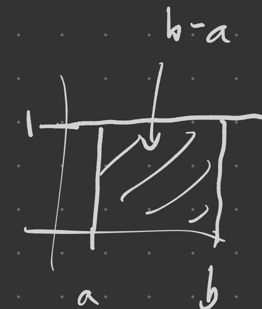
constant!

A

$$\int_a^b \left(f(x) - f_{ave}\right) dx = \int_a^b f(x)\,dx - \int_a^b f_{ave}\,dx$$

$$= \int_a^b f(x)\,dx - f_{ave}\int_a^b 1\,dx$$

$$= \int_a^b f(x)\,dx - (b-a)\cdot f_{ave}$$



$$= \int_a^b f(x)\,dx - \int_a^b f(x)\,dx$$

$$= 0.$$

$$\int_a^b c\,dx$$

$$= c\cdot(b-a)$$