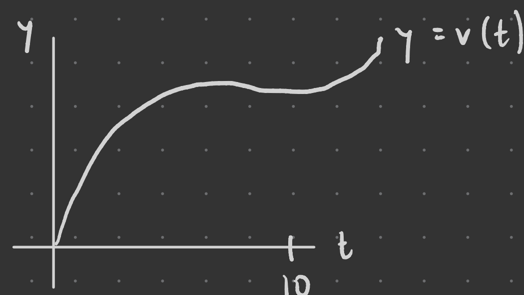
24. X. 28

Goals

- Motivate integration
- Sigma (Σ) notation
- Riemann sums

E.g. Suppose object travels east with velocity $v(t)$ m/s at time t seconds. What is the change in the object's position from $t=0$ to $t=10$?



$\rightsquigarrow$ $s(10) - s(0) = ?$

$s(t)$ = pos'n at time t (m)

If $v(t) = c$ is constant, this is simple:

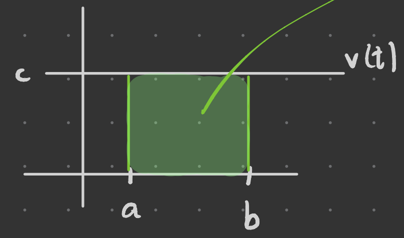
distance = const velocity × time

so $s(10) - s(0) = c \cdot (10 - 0)$.

More generally,

$$s(b) - s(a) = c \cdot (b - a) \quad \text{for } a \le b.$$

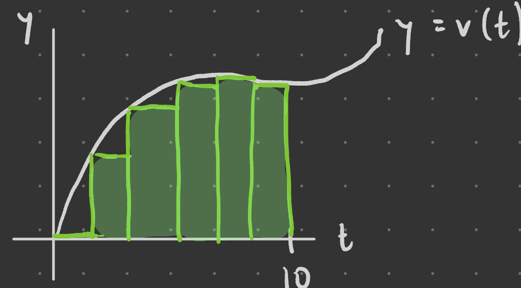
Geometrically:

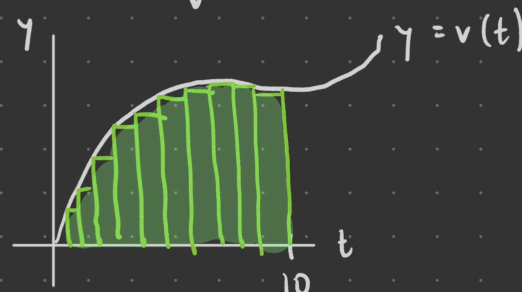


area = height × width

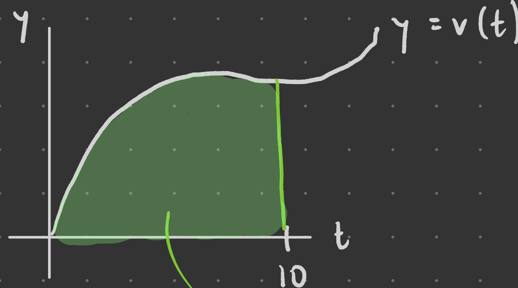
$= c \cdot (b - a)$

Divide and conquer: Approximate displacement by pretending $v(t)$ is constant on small intervals:





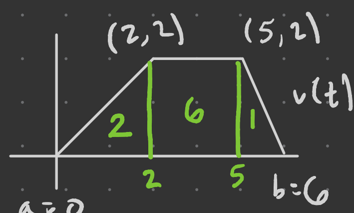
And "in the limit"



this area equals $s(10) - s(0)$.

It is also "$\int_0^{10} v(t)\,dt$".

Problems For the following velocity graphs, find the displacement (total change in position) from $t = a$ to $t = b$:



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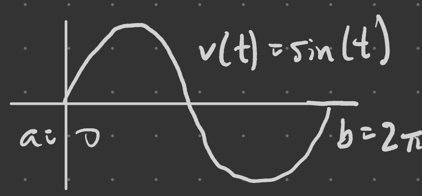

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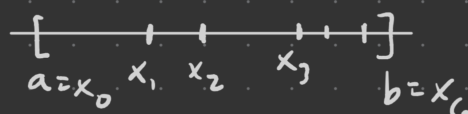
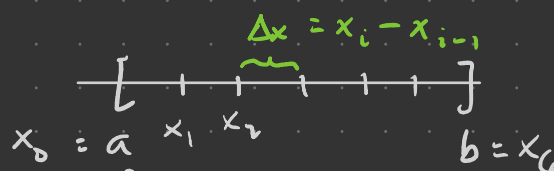
Let's formalize:

Defn Fix $a \le b$. A set of points $P = \{x_0, x_1, \ldots, x_n\}$ with $a = x_0 < x_1 < x_2 < \cdots < x_n = b$ is called a *partition* of $[a,b]$.
If the subintervals all have the same width, the set of points forms a *regular partition* of the interval $[a,b]$.

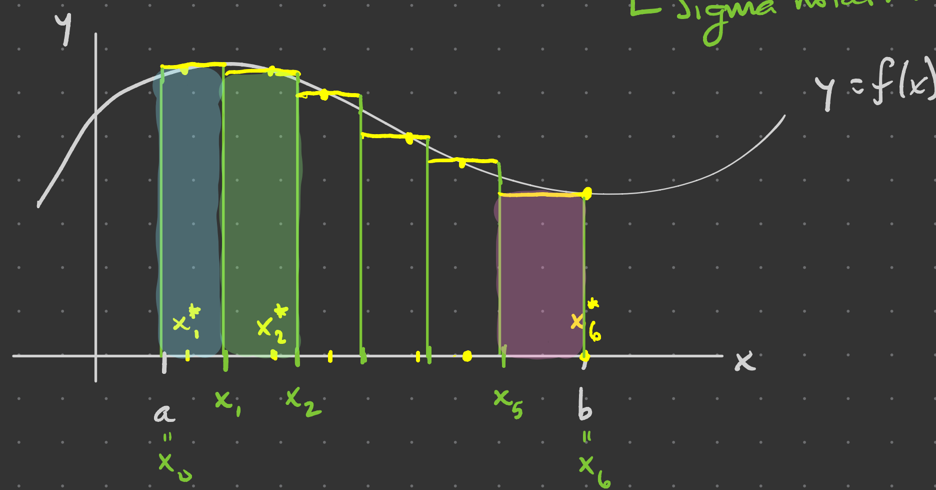




Defn Let $f : [a,b] \to \mathbb{R}$ be a function, $P = \{x_0, x_1, \ldots, x_n\}$ be a regular partition of $[a,b]$ with subinterval width $\Delta x = x_i - x_{i-1}$. For each i, let x_i^* be some point of $[x_{i-1}, x_i]$. The *Riemann sum* for f over $[a,b]$ relative to P, $\{x_i^*\}$ is

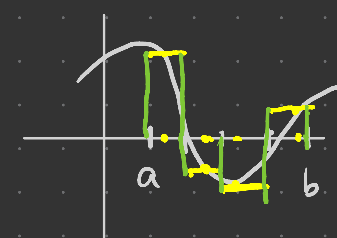
$$f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x = \sum_{i=1}^{n} f(x_i^*)\,\Delta x_i .$$

↳ Sigma notation — more soon!



Flavors of Riemann sum:

Left, right, midpoint, upper, lower.



Note The "area" $f(x_i^*)\,\Delta x$ is negative when $f(x_i^*) < 0$.

⇝ "signed area"

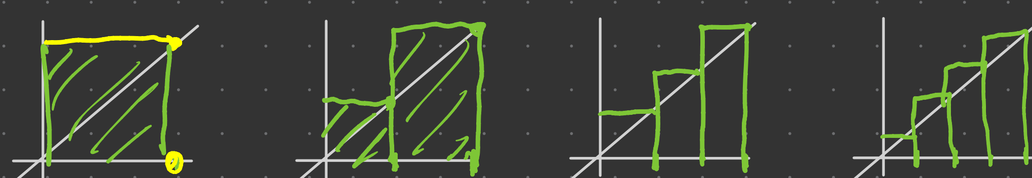
Upshot Signed area under curve between a, b is

$$\approx f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x$$

(especially for n large ⇒ Δx small).

If $f(t) = v(t)$ is velocity, this is ≈ displacement from a to b.

Problem Find right Riemann sum for $f(x) = x$ on $[0,1]$ with $n = 1, 2, 3, 4$.



1

$\frac{1}{2}\cdot\frac{1}{2} + 1\cdot\frac{1}{2} = \frac{3}{4}$

$\frac{1}{3}\cdot\frac{1}{3} + \frac{2}{3}\cdot\frac{1}{3} + 1\cdot\frac{1}{3} = \frac{2}{3}$

$\frac{1}{4}\cdot\frac{1}{4} + \frac{1}{2}\cdot\frac{1}{4} + \frac{3}{4}\cdot\frac{1}{4} + 1\cdot\frac{1}{4} = \frac{5}{8}$

Sigma notation

Σ : Greek S for Sum.

Sequence $(a_i)_{i=1}^{\infty} = (a_1, a_2, a_3, a_4, \ldots)$

end

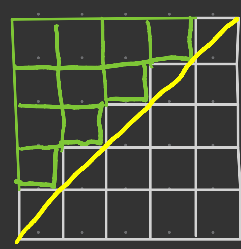
$$\sum_{i=1}^{n} a_i := a_1 + a_2 + a_3 + \cdots + a_n = \sum_{1 \le i \le n} a_i$$

start

$$\sum_{i=3}^{5} \sin(\pi i) = \sin(\pi \cdot 3) + \sin(\pi \cdot 4) + \sin(\pi \cdot 5)$$

E.g. If $a_i = i$, then $\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} i = 1 + 2 + 3 + \cdots + n$

is the sum of the first n positive integers.



$n \times n$ square

n^2 total squares

$$\frac{1}{2}n^2 + \frac{1}{2}n = \frac{n^2+n}{2} = \frac{n(n+1)}{2}$$

Thus $$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}.$$

Check

$$1 = \frac{1\cdot 2}{2}$$

$$1+2 = 3 = \frac{2\cdot 3}{2}$$

$$1+2+3 = 6 = \frac{3\cdot 4}{2}$$

$$1+2+3+4 = 10 = \frac{4\cdot 5}{2}$$

$$1+2+3+4+5 = 15 = \frac{5\cdot 6}{2}$$ ✓

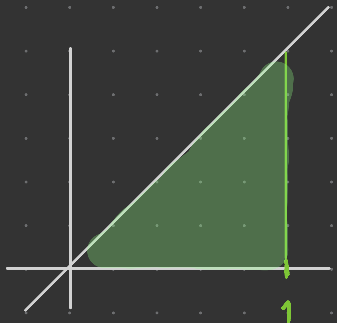
Let's return to the right hand Riemann sum for $f(x) = x$ over $[0,1]$:

$$R_n = \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

n-th right Riemann sum

$$= \sum_{i=1}^{n} \frac{i}{n} \cdot \frac{1}{n}$$

$$= \frac{1}{n^2} \sum_{i=1}^{n} i \qquad \text{[factor out } \tfrac{1}{n^2}\text{]}$$



$$= \frac{1}{n^2} \cdot \frac{n(n+1)}{2}$$

$f(x) = x$

$$= \frac{n+1}{2n} \xrightarrow[n \to \infty]{} \frac{1}{2} = \int_0^1 x \, dx$$

n	1	2	3	4
$\frac{n+1}{2n}$	$\frac{2}{2}$	$\frac{3}{4}$	$\frac{4}{6} = \frac{2}{3}$	$\frac{5}{8}$

$$R_{500} = \frac{501}{1000}$$