Goals

- Mean value theorem for integrals
- State FTC1 & understand why it's true
- Use FTC1 to evaluate derivatives of accumulation functions

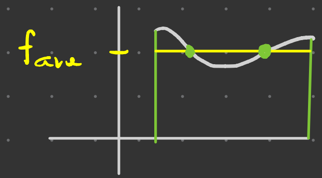
Mean value theorem: $f : [a,b] \to \mathbb{R}$ cts on $[a,b]$, diff'l on (a,b).
Then there exists c in (a,b) such that

$$f'(c) = \frac{f(b) - f(a)}{b-a}.$$

Slogan Instantaneous rate of change at some point is equal to avg rate of change.

Mean value theorem for integrals If $f:[a,b]\to\mathbb{R}$ is continuous, then there exists c in $[a,b]$ such that

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$



Slogan A continuous function takes its average value at least once.

Proof By the extreme value theorem, f attains its min and max values m, M on $[a,b]$. Then $m \le f(x) \le M$ on $[a,b]$, so

$$m\cdot(b-a) \le \int_a^b f(x)\,dx \le M\cdot(b-a)$$

$$\Rightarrow \quad m \le \frac{1}{b-a}\int_a^b f(x)\,dx \le M.$$

Since $\frac{1}{b-a}\int_a^b f(x)\,dx$ is between m, M

+ f takes the values m, M

+ f continuous

the intermediate value theorem implies there is some c for which

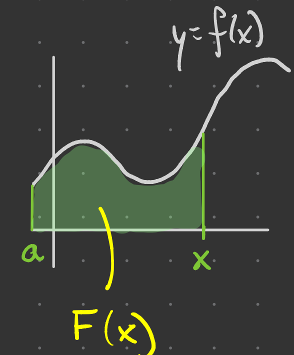
$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx. \quad \square$$

FTC 1 If $f:[a,b] \to \mathbb{R}$ is continuous, and

$$F(x) := \int_a^x f(t)\,dt,$$

← accumulation function of f

then $F'(x) = f(x)$.



Call $F(x) = \int_a^x f(t)\,dt$ an accumulation function.

Slogan $\overbrace{\text{Accumulation function of } f}^{F}$ is an antiderivative of f.

antiderivative of f is a function F such that $F' = f$

Pf We have

$$F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h}$$

$$= \lim_{h \to 0} \frac{1}{h} \left(\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt \right)$$

$$= \lim_{h \to 0} \underbrace{\frac{1}{h} \int_x^{x+h} f(t)\,dt}_{\text{average of } f \text{ over } [x,\, x+h]}$$

$\left[\int_a^b = \int_a^c + \int_c^b \right]$

By MVT4$\int$, there is some $c \in [x, x+h]$ such that

$$f(c) = \frac{1}{h}\int_x^{x+h} f(t)\,dt.$$

As $h \to 0$, $c \to x$. Since f is continuous,

$$\lim_{h\to 0} f(c) = \lim_{c \to x} f(c) = f(x).$$

Thus $F'(x) = \lim_{h\to 0} \frac{1}{h}\int_x^{x+h} f(t)\,dt$

$= \lim_{h \to 0} f(c)$

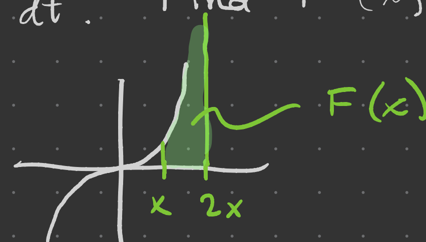
$= f(x)$ as desired. $\square$

E.g. If $g(x) = \int_1^x \frac{1}{t^3+1}\,dt$, then $g'(x) = \frac{1}{x^3+1}$

E.g. If $F(x) = \int_1^{x^3} \cos t\,dt$, then we can use the chain rule to compute $F'(x)$. Let $u(x) = x^3$. Then $F(x) = \int_1^{u(x)} \cos t\,dt$

and
$$F'(x) = \cos(u(x)) \cdot u'(x)$$
$$= \cos(x^3) \cdot 3x^2 .$$

Problem Let $F(x) = \int_x^{2x} t^3\,dt$. Find $F'(x)$.



Answer

$$F(x) = \int_0^{2x} t^3\,dt - \int_0^{x} t^3\,dt$$

so

$$F'(x) = \underbrace{16x^3}_{?} - x^3 = 15x^3$$

$$\int_0^{2x} t^3\,dt = g(2x) \quad \text{for} \quad g(x) = \int_0^{x} t^3\,dt$$

thus

$$\frac{d}{dx}\int_0^{2x} t^3\,dt = g'(2x)\cdot(2x)'$$

$$= (2x)^3 \cdot 2 = 16x^3$$

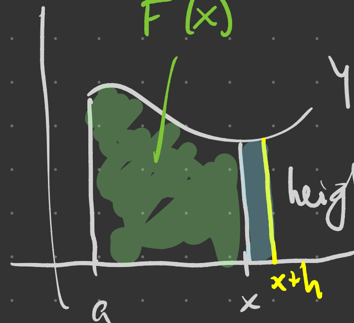
Note If $G(x) = \int_a^{u(x)} f(t)\,dt$, then

$$G'(x) = f(u(x)) \cdot u'(x).$$

(via chain rule).

(1) Informal proof of FTC1

(2) Continuity hypothesis

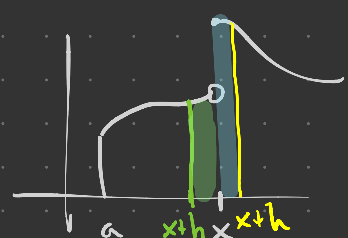


$$F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h}$$

blue area

$$\frac{F(x+h) - F(x)}{h} \approx \frac{h \cdot f(x)}{h} = f(x)$$

What if f has discontinuity?



so $\lim_{h \to 0} \frac{F(x+h) - F(x)}{h}$ DNE