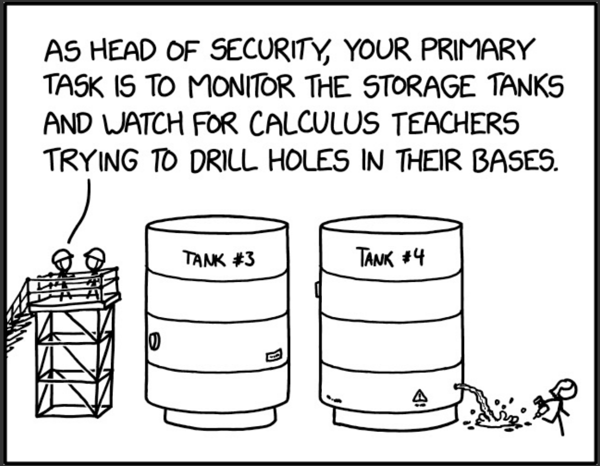
24. X. 9

Related Rates

Idea: Implicit differentiation relates the rates of change of quantities appearing together in an equation.

We can use this to solve cool problems.



xkcd 2974

E.g. Spherical balloon inflating at a constant rate of 2 cm^3/sec.

How fast is the radius increasing when the radius is 6 cm?

Know $V = \frac{4}{3}\pi r^3$, with both V and r functions of time t.

Differentiating with respect to t:

$$\frac{dV}{dt} = \frac{4}{3}\pi \cdot 3r^2 \cdot \frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}$$

$$\frac{dV}{dt} = 2 \text{ cm}^3/\text{sec}$$

We are told $\frac{dV}{dt} = 2 \text{ cm}^3/\text{sec}$, so

$$2 = 4\pi r^2 \frac{dr}{dt} \implies \frac{dr}{dt} = \frac{1}{2\pi r^2}.$$

Thus $$\left.\frac{dr}{dt}\right|_{r=6\text{ cm}} = \frac{1}{2\pi \cdot 6^2} \text{ cm/sec} = \frac{1}{72\pi} \text{ cm/sec}.$$ ✓

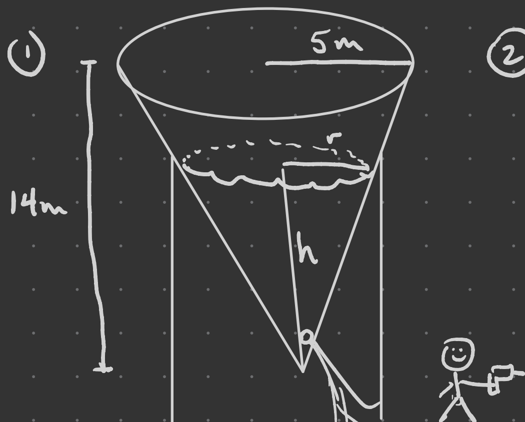
General Strategy

1. Draw and label
2. State given info + rate to be determined
3. Find an equation relating the variables
4. Implicitly differentiate
5. Substitute known info from 2 and solve for desired rate — include units!

E.g. A conical tank of water has a height of 14 meters and radius of 5 meters at the top. Your calculus teacher has drilled a hole into the bottom of the tank and it is leaking water at a rate of 10 cm^3/sec.

How quickly is the height of the water in the tank changing when its height is 5 meters?

(1)



(2) Know $\dfrac{dV}{dt} = -10 \ cm^3/sec$

neg b/c vol decreasing

Want $\left.\dfrac{dh}{dt}\right|_{h=5m}$

(3) $V = \frac{1}{3}\pi r^2 h$

is the volume of water

By similar triangles, $\dfrac{5}{14} = \dfrac{r}{h} \Rightarrow r = \dfrac{5h}{14}$ m

and $V = \frac{1}{3}\pi\left(\frac{5}{14}\right)^2 h^2 h = \dfrac{5\pi}{588} h^3$

④ $$\frac{dV}{dt} = \frac{5\pi}{588} 3h^2 \frac{dh}{dt} = \frac{15\pi}{588} h^2 \frac{dh}{dt}.$$

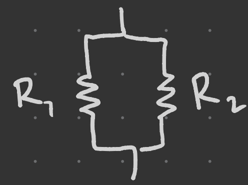
⑤ Know $$\frac{dV}{dt} = -10 \text{ cm}^3/\text{sec} = -10 \cdot \left(\frac{1}{100} \frac{\text{m}}{\text{cm}}\right)^3 \cdot \frac{\text{cm}^3}{\text{sec}}$$

$$= -10^{-5} \frac{\text{m}^3}{\text{sec}}.$$

So $$-10^{-5} = \frac{15\pi}{588} h^2 \frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{-588}{15\pi \cdot 10^5 h^2}$$

and $$\left.\frac{dh}{dt}\right|_{h=5\text{ m}} = \frac{-588}{15\pi \cdot 10^5 \cdot 25} \text{ m/sec} \approx -4.99 \cdot 10^{-6} \text{ m/sec}.$$

Problem 1



total resistance R satisfies

Ohm's law $\dfrac{1}{R} = \dfrac{1}{R_1} + \dfrac{1}{R_2}$.

Suppose R_1 increasing at a rate of 0.4 Ohms/sec
R_2 decreasing at a rate of 0.5 Ohms/sec

At what rate is R changing when $R_1 = 100$ Ohms, $R_2 = 111$ Ohms?

Problem 2

Person A standing at $(0,0)$ begins walking north at a rate of 20 units/sec; person B standing at $(0,50)$ begins walking west at a rate of 10 units/sec. How quickly is the distance between them changing when they are 70 units apart?

Problem 3 Boyle's law says $PV = c$ in a gas with constant temperature where $P =$ pressure, $V =$ volume, and c is a constant.

Suppose a gas is in a cylinder with piston and its initial volume is $250\ cm^3$, pressure $100\ kPa$. The piston is depressed so that volume decreases at a rate of $50\ cm^3/min$. How quickly will the pressure of the gas initially increase?