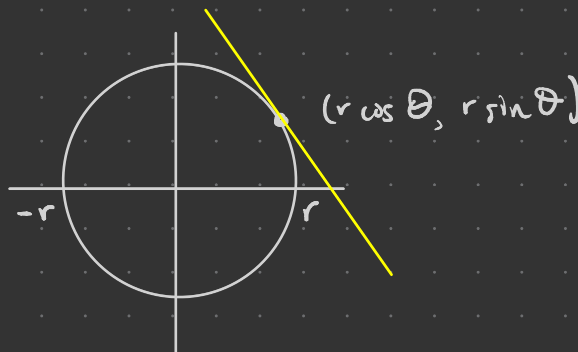
Implicit Differentiation

Motivation Suppose we want to find tangent lines to a radius r circle centered at the origin.



This circle has equation

$$x^2 + y^2 = r^2$$

so $y = \begin{cases} \sqrt{r^2 - x^2} \\ -\sqrt{r^2 - x^2} \end{cases}$

Multi-valued so not a function!

Treat y as a "local" function of x to find derivatives.

① Differentiate both sides of the equation, using the chain rule to treat y as a function of x

② Solve for $\frac{dy}{dx}$:

(a) Put $\frac{dy}{dx}$ terms on left side of eq'n, factor out $\frac{dy}{dx}$

(b) Solve for $\frac{dy}{dx}$ by dividing both sides by other factor on left.

E.g. $$\frac{d}{dx}\left(x^2 + y^2 = r^2\right)$$

$\Big\downarrow d/dx$ —— chain rule!

$$2x + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow \quad 2y\frac{dy}{dx} = -2x$$

for $y \neq 0$

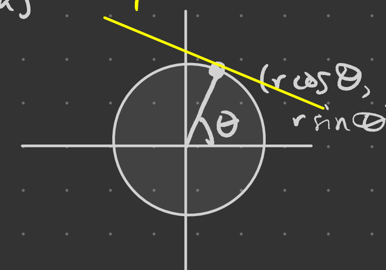
$$\Rightarrow \quad \frac{dy}{dx} = \frac{-x}{y}$$

Depends on x and y!
Typical for implicit differentiation

So the tangent line through $(r\cos\theta, r\sin\theta)$ has

slope $\dfrac{-r\cos\theta}{r\sin\theta} = -\cot\theta$.

slope $-\dfrac{1}{\tan\theta}$



E.g. Find $\frac{dy}{dx}$ for the curve given by $x^3\sin y + y = 4x+3$:

$$\frac{d}{dx}(x^3\sin y) + \frac{dy}{dx} = 4$$

$$3x^2\sin y + x^3\cos y\frac{dy}{dx} + \frac{dy}{dx} = 4$$

$$(x^3\cos y + 1)\frac{dy}{dx} = 4 - 3x^2\sin y$$

$$\frac{dy}{dx} = \frac{4-3x^2\sin y}{x^3\cos y + 1}$$

so slope of tangent at $(0,3)$ is $\dfrac{4-0}{0+1} = 4$

Problem Find $\frac{dy}{dx}$ when $xy\cos(xy) + 1 = 0$.

Derivatives of logs

Recall $\log(x)$ is inverse to e^x (i.e. $\log = \log_e = \ln$)

$\log_b(x)$ is inverse to b^x

Then for $y = \log x$, $e^y = x$. Differentiating both sides,

for $x > 0$

$$e^y\frac{dy}{dx} = 1$$

$$\Rightarrow x \frac{dy}{dx} = 1$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{x} \quad \text{for } x > 0$$

i.e. $\boxed{\log'(x) = \frac{1}{x}}$ for $x > 0$.

If $y = \log_b x$, then $b^y = x$. Applying log to both sides,

$$y \log b = \log x$$

$$y = \frac{\log x}{\log b}$$

Thus $\boxed{\log_b'(x) = \frac{1}{x \log b}}$ for $x > 0$.

$\log_e b$

If $y = b^x$, then $\log y = x \log b$. Differentiating,

$$\frac{1}{y}\frac{dy}{dx} = \log b$$

$$\Longrightarrow \frac{dy}{dx} = y \log b$$

i.e. $\boxed{\frac{d}{dx} b^x = (\log b)\, b^x}$.

We just employed a new technique!: Logarithmic differentiation:

$$y = f(x)^{g(x)} \Longrightarrow \log y = g(x) \log f(x)$$

$$\Longrightarrow \frac{1}{y}\frac{dy}{dx} = g'(x) \log f(x) + \frac{g(x)}{f(x)} f'(x)$$

$$\Longrightarrow \quad \frac{dy}{dx} = f(x)^{g(x)} \cdot \left(g'(x) \log f(x) + \frac{g(x)}{f(x)} f'(x) \right)$$

E.g. $y = x^x \Longrightarrow \log y = x \log x$

$$\frac{1}{y}\frac{dy}{dx} = \log x + \frac{x}{x} = 1 + \log x$$

$$\frac{dy}{dx} = x^x (1 + \log x).$$