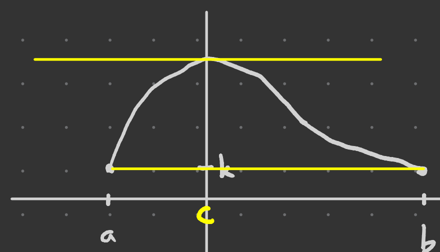
24.X.11

Goals
- Rolle's theorem
- Mean value theorem
- Sign of $f'(x)$ detects increasing/decreasing

Rolle's Thm Let $f:[a,b]\to\mathbb{R}$ be continuous, differentiable over (a,b), and such that $f(a)=f(b)$. Then there is at least one c in (a,b) such that $f'(c)=0$.



Pf Let $k=f(a)=f(b)$.

Case 1: $f(x)=k$ for all x in (a,b)

Then $f'(x)=0$ for all x in (a,b) ✓

Case 2: There exists x in (a,b) such that $f(x)>k$.

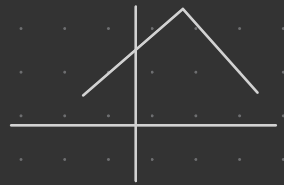
By the extreme value theorem, f has an absolute max on $[a,b]$, and this occurs at some c in (a,b).

By Fermat's theorem, c is a critical point, so $f'(c)=0$. ✓

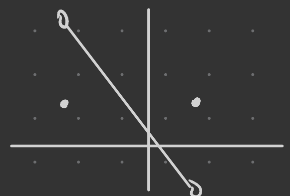
<u>Case 3</u>: There exists x in (a,b) such that $f(x)<k$.

Use a similar argument! □

<u>Note</u> Must have f diff'l for Rolle to apply:



Also must have f continuous:

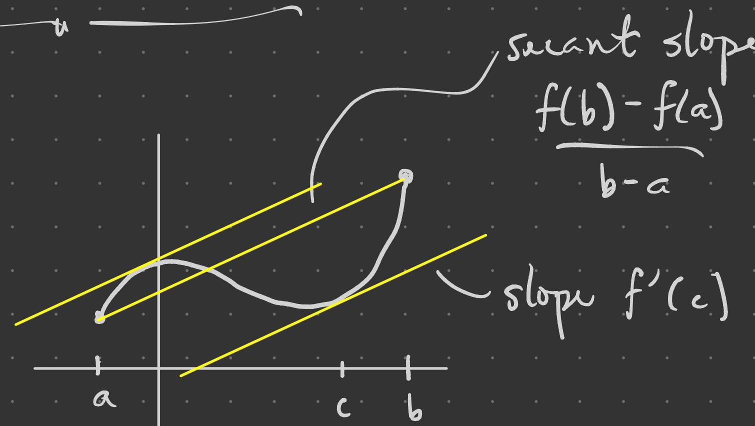


E.g. Consider $f(x) = x^3 - 9x = x(x^2-9) = x(x+3)(x-3)$.

Since $f(0) = f(-3) = f(3) = 0$, know f has a critical point in $(-3,0)$ and in $(0,3)$ by Rolle.

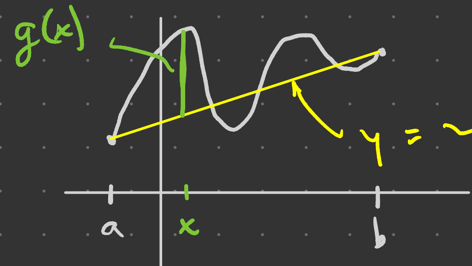
Check: $f'(x) = 3x^2 - 9 = 3(x^2-3) = 3(x+\sqrt{3})(x-\sqrt{3})$ has roots at $x = \pm\sqrt{3}$ ✓

What if $f(a) \neq f(b)$?



Mean value theorem Let f be continuous over $[a,b]$, diff'l over (a,b). Then there exists at least one c in (a,b) such that
$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$

Pf Set
$$g(x) = f(x) - \left(\frac{f(b)-f(a)}{b-a}(x-a) + f(a) \right)$$



Then g is continuous on $[a,b]$, diff'l on (a,b), and

$g(a) = g(b) = 0$. By Rolle, there exists c in (a,b) with

$g'(c) = 0$. But $g'(x) = f'(x) - \dfrac{f(b)-f(a)}{b-a}$

so $\quad 0 = g'(c) = f'(c) - \dfrac{f(b)-f(a)}{b-a}$

$$\Rightarrow f'(c) = \frac{f(b)-f(a)}{b-a}. \qquad \square$$

Application If a car's average speed exceeds the speed limit, then there is at least one moment when the car's instantaneous speed exceeds the speed limit.

corollary *noun*

cor·ol·lary ˈkȯr-ə-ˌler-ē ˈkär-, -le-rē, *British* kə-ˈrä-lə-rē

plural **corollaries**

Synonyms of *corollary* >

1 : a proposition (see PROPOSITION entry 1 sense 1c) inferred immediately from a proved proposition with little or no additional proof

2 a : something that naturally follows : **RESULT**

… love was a stormy passion and jealousy its normal *corollary*.
– Ida Treat

b : something that incidentally or naturally accompanies or parallels

A *corollary* to the problem of the number of vessels to be built was that of the types of vessels to be constructed.
– Daniel Marx

Three corollaries of MVT:

Cor 1 Let f be diff'l over an interval I. If $f'(x)=0$ for all x in I, then $f(x)$ is constant on I.

"If you don't change, you stay the same!"

Pf Suppose f is not constant and take $a<b$ with $f(a)\neq f(b)$.

Then $\frac{f(b)-f(a)}{b-a}\neq 0 \Rightarrow$ there is some c in I with $f'(c)=\frac{f(b)-f(a)}{b-a}\neq 0$, contradicting hypothesis that $f'=0$. $\square$

Cor 2 If f,g diff'l on an interval I and $f'(x)=g'(x)$ for all x in I, then $f(x)=g(x)+C$ for some constant C.

Pf Apply Cor 1 to $f(x)-g(x)=h(x)$. Indeed, $h'(x)=f'(x)-g'(x)=0 \Rightarrow h(x)=C \Rightarrow f(x)=g(x)+C$. □

Cor 3 Let f be continuous on $[a,b]$, diff'l on (a,b).

(i) If $f'(x)>0$ for x in (a,b), then f is increasing on $[a,b]$.

(ii) If $f'(x)<0$ for x in (a,b), then f is decreasing on $[a,b]$.

Pf of (i) For contradiction, suppose f not increasing on I, so there exist $a<b$ in I with $f(b)<f(a)$.

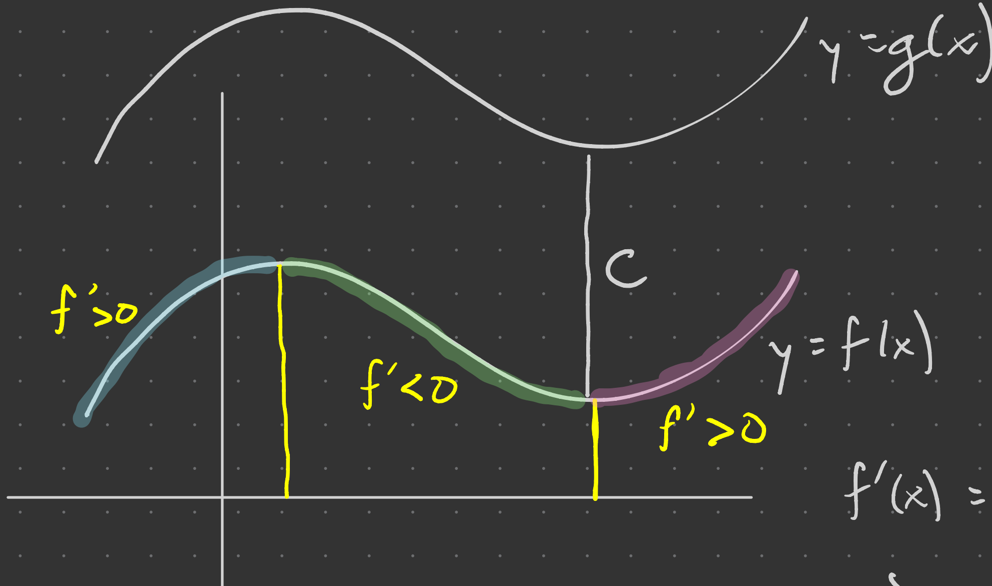
By MVT, there exists c in (a,b) with

$$f'(c) = \frac{f(b)-f(a)}{b-a} < 0$$

a contradiction! □

Increasing: $c<d \Rightarrow f(c)<f(d)$

Decreasing: $c<d \Rightarrow f(c)>f(d)$



$$f'(x) = g'(x)$$

$$\Rightarrow \quad g(x) = f(x) + c$$