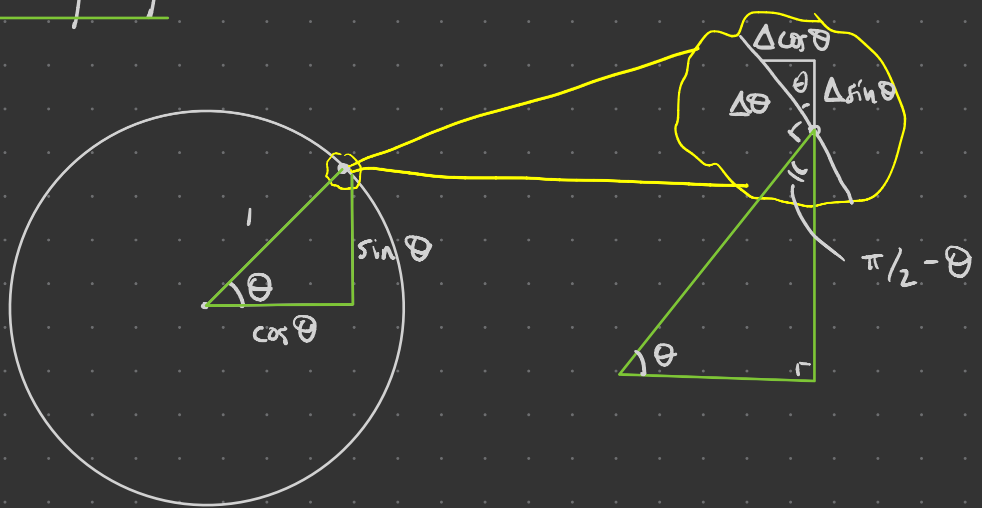
24. IX. 29

Goals • Differentiate trig functions, exponentials, and logarithms

Thm $\frac{d}{dx}\sin x = \cos x$ and $\frac{d}{dx}\cos x = -\sin x$.

See text for derivation via limits and angle addition formula.

Geometric proof



$$\frac{d}{d\theta}\sin\theta \approx \frac{\Delta \sin\theta}{\Delta\theta} \approx \cos\theta$$

$$\frac{d}{d\theta}\cos\theta \approx \frac{\Delta \cos\theta}{\Delta\theta} \approx -\sin\theta$$

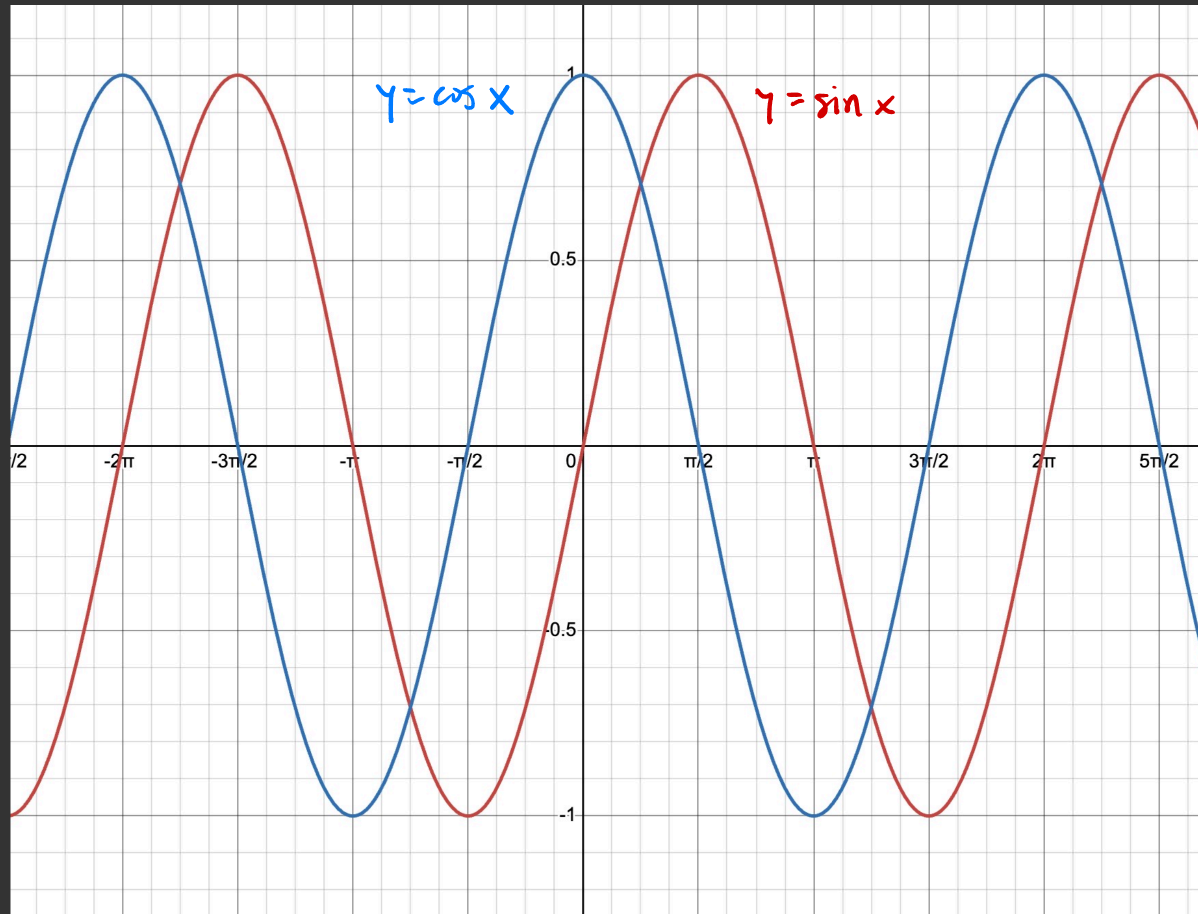
Observe:

$$(\sin x)' = \cos x$$

$$(\cos x)' = -\sin x$$

$$(-\sin x)' = -\cos x$$

$$\begin{aligned}(-\cos x)' &= -(-\sin x) \\ &= \sin x\end{aligned}$$



E.g. ① $(x^2 \cos x)' = (x^2)' \cos x + x^2 (\cos x)'$

$= 2x \cos x - x^2 \sin x$

② $\left(\dfrac{x}{\sin x}\right)' = \dfrac{\sin x \cdot (x)' - x(\sin x)'}{(\sin x)^2}$

$= \dfrac{\sin x - x \cos x}{\sin^2 x}$

③ $\left(\sin(x^3)\right)' = \sin'(x^3) \cdot (x^3)'$

$= \cos(x^3) \cdot 3x^2$

$= 3x^2 \cos(x^3)$

④ $$\left(\frac{\sin x}{\cos x}\right)' = \frac{\cos x\,(\sin x)' - \sin x\,(\cos x)'}{(\cos x)^2}$$

$$\parallel$$

$$(\tan x)'$$

$$= \frac{\cos x \cdot \cos x + \sin x \cdot \sin x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x}$$

$$= \frac{1}{\cos x} \cdot \frac{1}{\cos x} = \sec^2 x$$

$f(x)$	$f'(x)$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x = \frac{\sin x}{\cos x}$	$\sec^2 x$
$\csc x = \frac{1}{\sin x}$	$-\csc x \cot x$
$\sec x = \frac{1}{\cos x}$	$\sec x \tan x$
$\cot x = \frac{1}{\tan x}$	$-\csc^2 x$

Further assume B is differentiable at 0 and that there exists a base b such that $B'(0) = 1$.

Call this base e and write $\exp(x) = e^x$.

e = Euler's constant $\approx 2.718281828\ldots$

Thm Fix $b > 0$ and let $B(x) = b^x$. Then

$$B'(x) = \underbrace{B'(0)}_{\text{will determine later}} b^x .$$

Pf We have

$$B'(x) = \lim_{h \to 0} \frac{b^{x+h} - b^x}{h}$$

$$= \lim_{h \to 0} \frac{b^x \cdot b^h - b^x}{h}$$

$$= \lim_{h \to 0} b^x \cdot \frac{b^h - 1}{h}$$

$$= b^x \cdot \lim_{h \to 0} \frac{b^h - 1}{h} = b^x \cdot B'(0)$$

$b^h = B(h)$

$1 = b^0 = B(0)$

$\square$

<u>Cor</u> $\exp' = \exp$, i.e. $(e^x)' = e^x$.

<u>E.g.</u> $\left(e^{-1/x^2}\right)' = e^{-1/x^2} \cdot \left(\frac{-1}{x^2}\right)'$

with value 0 at $x = 0$

$$= e^{-1/x^2} \cdot \left(-x^{-2}\right)'$$

$$= e^{-1/x^2} \cdot \left(2x^{-3}\right)$$

$$= \frac{2e^{-1/x^2}}{x^3}.$$

24.X.2

Recall $\log_b = B^{-1}$ for $B(x) = b^x$.

Let $\log_e = \exp^{-1} =: \log$ (also denoted $\ln$).

Thm $(\log x)' = \frac{1}{x}$ for $x > 0$.

Pf By the inverse function theorem,