24. IX. 9

Goals

- Review composition & inverses
- Chain rule : $(f \circ g)'$
- Inverse function theorem : $(f^{-1})'$

Composition

$$\to \boxed{g} \to \;\circ\; \to \boxed{f} \to \;=\; \overset{f\circ g}{\boxed{\to \boxed{g} \to \boxed{f} \to}}$$

i.e. $(f \circ g)(x) = f(g(x))$ (as long as $g(x)$ is in the domain of f).

For instance, suppose $f(x) = x - 1$ and $g(x) = x^2$.

Then $(f \circ g)(x) = f(g(x))$

$= f(x^2)$

$$x \to \boxed{g} \to x^2 \to \boxed{f} \to x^2 - 1$$

$= x^2 - 1$

2. $f \circ g \neq g \circ f$ in general.

With above functions, $(g \circ f)(x) = g(f(x))$

$= g(x-1)$

$= (x-1)^2$

$= x^2 - 2x + 1$.

$x \xmapsto{z} x^2+5 \xmapsto{w} \frac{1}{x^2+5}$

$z(x) = x^2+5$

$w(x) = \frac{1}{x}$

$g = w \circ z$

Problem Express the following functions as compositions:

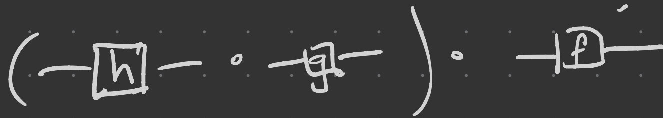
- $f(x) = \sin^2 x = (\sin x)^2$
- $g(x) = \frac{1}{x^2+5}$

$x \xmapsto{k} \sin x \xmapsto{l} (\sin x)^2$

$l(x) = x^2$

$k(x) = \sin x$

$f = l \circ k$



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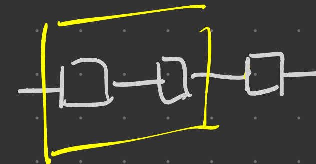
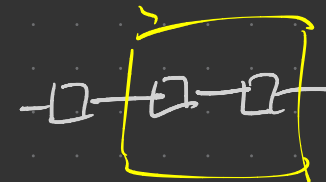


$$f \circ (g \circ h) = (f \circ g) \circ h$$



I.e. function composition is associative

If $f = g \circ h$, then $f(x) = (g \circ h)(x) = g(h(x))$

Inverses The function $id(x) = x$ is called the **identity function**.

Its job is to do nothing.

Defn If $f \circ g = id$ and $g \circ f = id$, call g the **inverse** of f and write $g = f^{-1}$.

$f^{-1} \neq \frac{1}{f}$

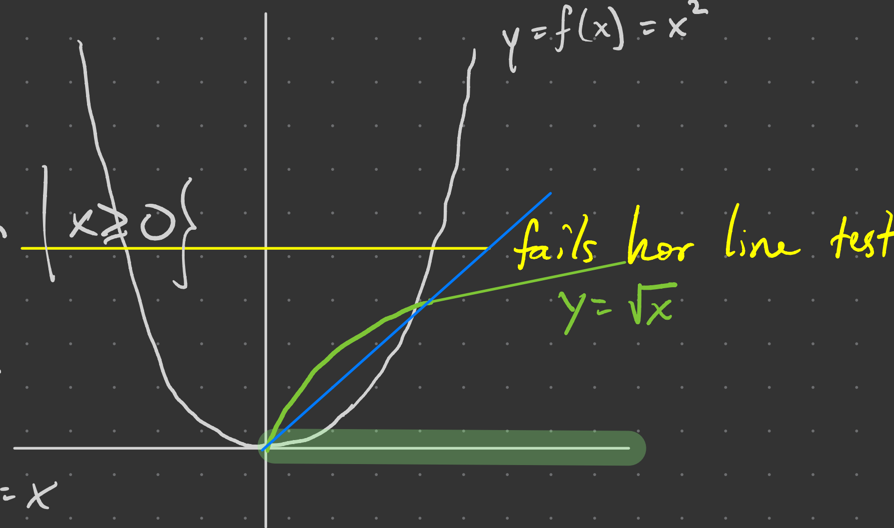
Note

- Many functions don't have an inverse
- Sometimes you need to restrict the domain of a function to specify an inverse
- The inverse is only defined on the image of f.
- If $f(x) = y$, then $(f^{-1} \circ f)(x) = f^{-1}(y)$

$\Rightarrow \quad x = f^{-1}(y)$ — inverses solve eq'ns!

E.g. $f(x) = x^2$

Restrict the domain of f to $\mathbb{R}_{\geq 0} = \{x \in \mathbb{R} \mid x \geq 0\}$



$g(x) = \sqrt{x}$ on $\mathbb{R}_{\geq 0}$.

Then $f(g(x)) = (\sqrt{x})^2 = x$

$g(f(x)) = \sqrt{x^2} = x$

Note Graph of $y = f^{-1}(x)$ is the reflection of $y = f(x)$ over $y = x$ line.

Chain Rule Let f, g be functions. Then for all x in the domain of g at which (1) g is diff'l, and (2) f is diff'l at $g(x)$,

$$\boxed{(f\circ g)'(x) = f'(g(x))\cdot g'(x).}$$

$y = g(x)$ $\quad$ $\dfrac{dz}{dx} = \dfrac{dz}{dy}\cdot\dfrac{dy}{dx}$

$z = f(y)$

E.g. Let's compute the derivative of $h(x) = (\underbrace{2x^3+5x^2-2x+3}_{})^4$.

Ah! This $h(x) = f(g(x))$ for $f(x) = x^4$, $g(x) = 2x^3+5x^2-2x+3$.

Thus
$$\begin{aligned} h'(x) &= f'(g(x))\cdot g'(x) \\ &= 4\cdot g(x)^3\cdot(6x^2+10x-2) \\ &= 4(2x^3+5x^2-2x+3)^3\cdot(6x^2+10x-2). \end{aligned}$$

Pf Idea

$$h'(a) = \lim_{x\to a} \frac{h(x)-h(a)}{x-a}$$ [defn]

$$= \lim_{x\to a} \frac{f(g(x))-f(g(a))}{x-a}$$ [$h = f\circ g$]

$$= \lim_{x\to a} \frac{f(g(x))-f(g(a))}{g(x)-g(a)} \cdot \frac{g(x)-g(a)}{x-a}$$ [mult by $1 = \frac{g(x)-g(a)}{g(x)-g(a)}$]

$$\boxed{= \lim_{x\to a} \frac{f(g(x))-f(g(a))}{g(x)-g(a)}} \cdot \lim_{x\to a} \frac{g(x)-g(a)}{x-a}$$ [prod rule for limits]

$$\boxed{= \lim_{y\to g(a)} \frac{f(y)-f(g(a))}{y-g(a)}} \cdot g'(a)$$ [$y = g(x)$ + as $x\to a$, $g(x)\to g(a)$] b/c g cts

defn of deriv

$$= f'(g(a))\cdot g'(a).$$ □ [defn]

Let's use the chain rule to derive the power rule for negative powers from positive powers:

Fix $n > 0$ and set $h(x) = x^{-n} = \frac{1}{x^n} = f(g(x))$ for $g(x) = x^n$, $f(x) = \frac{1}{x}$.

We know $g'(x) = n x^{n-1}$, $f'(x) = \frac{-1}{x^2}$. Thus

$$h'(x) = \frac{-1}{(x^n)^2} \cdot n x^{n-1} \qquad \text{[chain rule]}$$

$$= \frac{-n}{x^{n+1}}$$

$$= -n x^{-n-1}$$ — still the power rule!

More generally, $\left(\frac{1}{g(x)}\right)' = \frac{-1}{g(x)^2} \cdot g'(x)$ [chain]

$$= \frac{-g'(x)}{g(x)^2}$$

and $\left(\frac{f(x)}{g(x)}\right)' = \left(f(x) \cdot \frac{1}{g(x)}\right)'$

$$= f'(x) \cdot \frac{1}{g(x)} + f(x) \cdot \left(\frac{-g'(x)}{g(x)^2}\right)$$

$$= \frac{f'(x)\,g(x) - f(x)\,g'(x)}{g(x)^2}$$

— the quotient rule!