24. IX. 18

**Goals** • Understand the following differentiation rules:

- Constant rule: For $c$ constant, $\boxed{\frac{d}{dx} c = 0}$
- Power rule: $\boxed{\frac{d}{dx} x^n = n x^{n-1}}$
- Linearity: For $c$ a constant, $f, g$ differentiable,

$$\boxed{\frac{d}{dx}(f + cg) = \frac{df}{dx} + c\frac{dg}{dx}}$$

- Product rule: For $f, g$ differentiable,

$$\boxed{\frac{d}{dx}\left(f(x) \cdot g(x)\right) = \frac{df}{dx} \cdot g(x) + f(x) \cdot \frac{dg}{dx}}$$

- Quotient rule: For $f, g$ differentiable, whenever $g(x) \neq 0$,

$$\boxed{\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\,\frac{df}{dx} - f(x)\cdot\frac{dg}{dx}}{g(x)^2}}$$

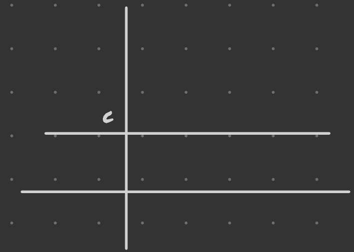
- Constant rule: For $c$ constant, $\boxed{\frac{d}{dx} c = 0}$

Pf For $f(x) = c$,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{c - c}{h}$$

$$= \lim_{h \to 0} 0 = 0. \quad \square$$



- Power rule: $\boxed{\frac{d}{dx} x^n = n x^{n-1}}$ for any constant $n$

We will justify the case where $n$ is a positive integer now, $n$ negative integer later, but the general case is true.

Pf for $n$ positive integer By the binomial theorem,

$$(x+h)^n = \underbrace{\binom{n}{0}}_{1} h^n + \underbrace{\binom{n}{1}}_{n} x h^{n-1} + \underbrace{\binom{n}{2}}_{\frac{n(n-1)}{2}} x^2 h^{n-2} + \cdots + \underbrace{\binom{n}{n-1}}_{n} x^{n-1} h + \underbrace{\binom{n}{n}}_{1} x^n$$

$$h \cdot \left( h^{n-1} + \binom{n}{1} x h^{n-2} + \cdots + \binom{n}{n-2} x^{n-2} h + n x^{n-1} \right)$$

so $$\lim_{h \to 0} \frac{(x+h)^n - x^n}{h} = \lim_{h \to 0} \left( h^{n-1} + \binom{n}{1} x h^{n-2} + \cdots + \binom{n}{n-2} x^{n-2} h + n x^{n-1} \right)$$

$= nx^{n-1}$. $\square$

E.g.

| $f(x)$ | $\frac{df}{dx}$ |
|---|---|
| $x = x^1$ | $1 \cdot x^0 = 1$ |
| $x^2$ | $2x^1 = 2x$ |
| $x^3$ | $3x^2$ |
| $x^4$ | $4x^3$ |
| $x^5$ | $5x^4$ |
| $x^6$ | $6x^5$ |
| $x^7$ | $7x^6$ |

But constant fractional and negative powers work too!

$$\frac{d}{dx}\sqrt[3]{x} = \frac{d}{dx} x^{1/3}$$

$$= \frac{1}{3} x^{-2/3} = \frac{1}{3\sqrt[3]{x^2}}$$

$$\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{d}{dx} x^{-1}$$

$$= -x^{-2} = \frac{-1}{x^2}$$

2 · Non-constant powers don't work! $\frac{d}{dx}(x^x) \neq x \cdot x^{x-1} = x^x$

$$\frac{d}{dx}(e^x) \neq xe^{x-1}$$

$= e^x$

- Linearity: For $c$ a constant, $f, g$ differentiable,

$$\boxed{\frac{d}{dx}(f + cg) = \frac{df}{dx} + c\frac{dg}{dx}}$$

In particular,

$$(f+g)' = f' + g' \qquad c=1$$

$$(f-g)' = f' - g' \qquad c=-1$$

$$(cf)' = c(f')$$

Pf $\quad \frac{d}{dx}\left(f + cg\right) = \lim_{h\to 0} \frac{f(x+h) + c\cdot g(x+h) - (f(x) + cg(x))}{h}$

$$= \lim_{h\to 0} \frac{(f(x+h) - f(x)) + c\cdot(g(x+h) - g(x))}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} + c \lim_{h\to 0} \frac{g(x+h) - g(x)}{h}$$

$$= \frac{df}{dx} + c\,\frac{dg}{dx} \qquad \square$$

E.g. We can differentiate polynomials!

$$(2x^3 - 5x^2 + x + 3)' = 2(x^3)' - 5(x^2)' + x' + (3)'$$

$$= 2 \cdot 3x^2 - 5 \cdot 2x^1 + 1 \cdot x^0 + 0$$

$$= 6x^2 - 10x + 1$$

Problem Determine $\frac{d}{dx}\left(\sqrt{x} - \frac{1}{\sqrt{x}}\right) = \frac{d}{dx}\left(x^{1/2} - x^{-1/2}\right)$

$$= \frac{d}{dx}\left(x^{1/2}\right) - \frac{d}{dx}\left(x^{-1/2}\right) = \frac{1}{2}x^{\frac{1}{2}-1} - \left(\frac{-1}{2}x^{-\frac{1}{2}-1}\right)$$

$$= \frac{1}{2}x^{-1/2} + \frac{1}{2}x^{-3/2} = \frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{x^3}}$$

- Product rule: For $f, g$ differentiable,

$$\boxed{\frac{d}{dx}\left(f(x)\cdot g(x)\right) = \frac{df}{dx}\cdot g(x) + f(x)\cdot\frac{dg}{dx}}$$

Pf Let $j(x) = f(x)\,g(x)$. Then

$$j'(x) = \lim_{h\to 0}\frac{j(x+h) - j(x)}{h}$$

$$= \lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0}\left(\frac{f(x+h) - f(x)}{h}g(x+h) + f(x)\frac{g(x+h) - g(x)}{h}\right)$$

$$\xrightarrow{\text{defn}} f'(x) \qquad \xrightarrow{g \text{ cts}} g(x) \qquad \xrightarrow{\lim(\text{const})} f(x) \qquad \xrightarrow{\text{defn}} g'(x)$$

$$= f'(x)g(x) + f(x)g'(x), \quad \text{as desired.} \quad \square$$

<u>E.g.</u>

$$(x^2)' = (x \cdot x)' = 1 \cdot x + x \cdot 1 = 2x$$

$$(x^3)' = (x^2 \cdot x)' = (2x) \cdot x + x^2 \cdot 1 = 3x^2$$

$$\vdots$$

$$(x^n)' = (x^{n-1} \cdot x)' = (n-1)x^{n-2} \cdot x + x^{n-1} \cdot 1$$

$$= (n-1)x^{n-1} + x^{n-1}$$

$$= n x^{n-1}$$

- Quotient rule: For $f, g$ differentiable, whenever $g(x) \neq 0$,

$$\boxed{\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\frac{df}{dx} - f(x)\cdot\frac{dg}{dx}}{g(x)^2}}$$

$$\frac{\text{low}\cdot d(\text{hi}) - \text{hi}\cdot d(\text{low})}{\text{low}^2}$$

We will give a quick proof of this after we see the chain rule!

E.g.

$$\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{x\cdot(1)' - 1(x')}{x^2}$$

$$= \frac{-1\cdot 1}{x^2}$$

$$= \frac{-1}{x^2}$$

**E.g.** If $h(x) = \dfrac{3x+1}{4x-3}$, then $h'(x) = \dfrac{(4x-3)\cdot 3 - (3x+1)\cdot 4}{(4x-3)^2}$

$$= \frac{12x - 9 - (12x+4)}{(4x-3)^2}$$

$$= \frac{-13}{(4x-3)^2}$$

**Problem** Where does $y = \dfrac{x}{x^2+1}$ have a horizontal tangent line?