24. IX. 11

Goals
- Limit laws
- Squeeze theorem
- Continuity

Limit laws f, g defined for $x \neq a$ in an open interval containing a, $\lim_{x\to a} f(x) = L$, $\lim_{x\to a} g(x) = M$. Take c a constant.

- $\lim_{x\to a} (f(x) + g(x)) = L + M$

  limit of a sum = sum of limits

- $\lim_{x\to a} c \cdot f(x) = cL$

- $\lim_{x\to a} f(x)g(x) = L \cdot M$

- $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}$ for $M \neq 0$

- $\lim_{x\to a} f(x)^n = L^n$ for all positive integers n

Q What about $n<0$

A $f(x)^{-2} = \dfrac{1}{f(x)^2}$ if $L=0$, L^{-2} DNE

- $\lim\limits_{x\to a} f(x)^{1/n} = L^{1/n}$ for all L if n is odd and for $L \geq 0$

  if n is even and $f(x) \geq 0$.

E.g. $\lim\limits_{x\to a} \sqrt{f(x)}$

$= \sqrt{\lim\limits_{x\to a} f(x)}$

for ≥ 0

Q Why the restrictions in the last law?

✓

E.g. $\lim\limits_{x\to 0} \dfrac{3x^2+x-5}{\sqrt{x+4}}$

Note $x+4 \xrightarrow[x\to 0]{} 4$ and $x+4 \geq 0$ near $x=0$, so $\sqrt{x+4} \xrightarrow[x\to 0]{} \sqrt{4} = 2$

Also $x^2 \xrightarrow[x\to 0]{} 0$, $x \xrightarrow[x\to 0]{} 0$, so $3x^2+x-5 \xrightarrow[x\to 0]{} 3\cdot 0 + 0 - 5 = -5$.

Thus $\lim\limits_{x \to 0} \dfrac{3x^2+x-5}{\sqrt{x+4}} = \dfrac{-5}{2}$.

i.e. p is continuous

<u>Thm</u> For p, q polynomials, $\lim\limits_{x \to a} p(x) = p(a)$

and $\lim\limits_{x \to a} \dfrac{p(x)}{q(x)} = \dfrac{p(a)}{q(a)}$ for $q(a) \neq 0$.

If $p(a) \neq 0$ and $q(a) = 0$, then $\lim\limits_{x \to a} \dfrac{p(x)}{q(x)}$ will be infinite or not exist;

if $p(a) = q(a) = 0$, then $\lim\limits_{x \to a} \dfrac{p(x)}{q(x)}$ may or may not exist.

E.g. $\lim_{x \to 2} \frac{x^2+x-6}{x-2}$

Note that $x^2+x-6 = (x-2)(x+3)$. Thus

$$\lim_{x \to 2} \frac{x^2+x-6}{x-2} = \lim_{x \to 2} \frac{\cancel{(x-2)}(x+3)}{\cancel{x-2}}$$

$$= \lim_{x \to 2} (x+3)$$

$$= 2+3 = 5.$$

If p is a polynomial, and $p(a)=0$, then

$$p(x) = (x-a) \cdot \underbrace{q(x)}_{\text{polynomial}}$$

$$x^2+x-6 = (x-2)(x+\underline{3}) \quad ✓$$

$\rightarrow ax^2+bx+c=0$

$$\Rightarrow x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

E.g. $\displaystyle\lim_{x\to -1} \frac{\sqrt{x+2}-1}{x+1}$ has indeterminate form $\frac{0}{0}$ at $x=-1$.

Trick: Multiply by $\dfrac{\sqrt{x+2}+1}{\sqrt{x+2}+1}$.

$$\boxed{\begin{aligned}&(y+a)(y-a)\\&= y^2-a^2\end{aligned}}$$

Doing so gives

$$\lim_{x\to -1} \frac{\sqrt{x+2}-1}{x+1} = \lim_{x\to -1}\left(\frac{\sqrt{x+2}-1}{x+1}\,\frac{\sqrt{x+2}+1}{\sqrt{x+2}+1}\right)$$

$$= \lim_{x\to -1} \frac{\cancel{x+2-1}}{\cancel{(x+1)}(\sqrt{x+2}+1)}$$

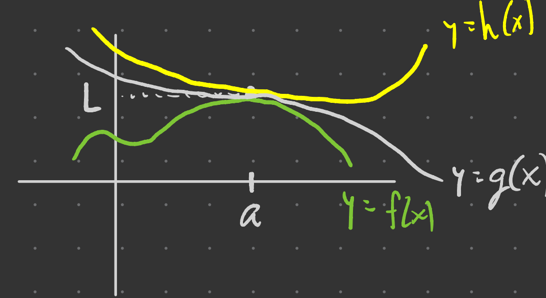
$$= \lim_{x\to -1} \frac{1}{\sqrt{x+2}+1} = \frac{1}{1+1} = \frac{1}{2}.$$

Problem Evaluate $\lim_{x \to 1} \frac{x+2}{(x-1)^2}$ by thinking about

$$\frac{x+2}{(x-1)^2} = \underbrace{(x+2)}_{\downarrow \atop 3} \cdot \underbrace{\frac{1}{(x-1)^2}}_{\downarrow \atop +\infty}.$$

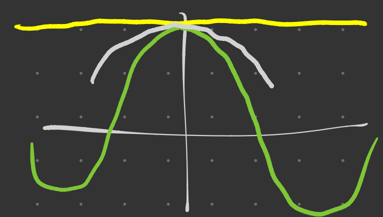
A $\lim_{x \to 1} \frac{x+2}{(x-1)^2} = +\infty$

Squeeze Theorem



If $f(x) \le g(x) \le h(x)$ near $x = a$ and

$\lim_{x \to a} f(x) = L = \lim_{x \to a} h(x)$, then $\lim_{x \to a} g(x) = L$.

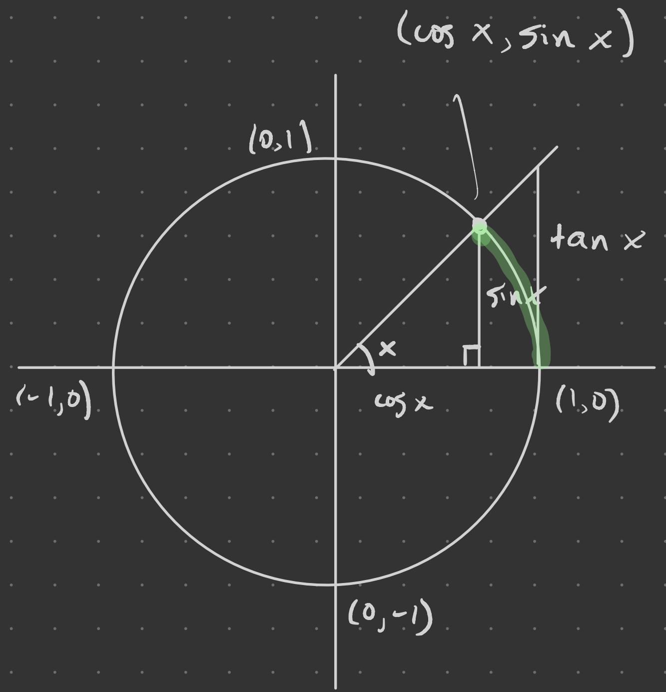


$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

By the diagram, for

$0 \le x \le \frac{\pi}{2}$

$0 \le \sin x \le x \cancel{\le \tan x}$



x

arclength of sector of unit circle is x, then angle is x radians

Divide by x:

$$\frac{0}{x} \le \frac{\sin x}{x} \le \frac{x}{x}$$

i.e. $0 \le \dfrac{\sin x}{x} \le 1$

Claim $\underset{\overset{x \to 0\ \downarrow}{1}}{\cos x} \le \underset{\overset{\downarrow \text{ by squeeze}}{1}}{\dfrac{\sin x}{x}} \le \underset{\overset{\downarrow\ x \to 0}{1}}{1}$

Got confused here!

Fix

As observed, $\sin x \le x \le \tan x$.

Dividing by $\sin x$:

$$1 \le \frac{x}{\sin x} \le \frac{\sin x}{\cos x} \cdot \frac{1}{\sin x} = \frac{1}{\cos x}$$

Inverting everything (which switches order):

$$\cos x \le \frac{\sin x}{x} \le 1.$$

Now squeeze! $\cos x \xrightarrow[x \to 0^+]{} 1$ and $1 \xrightarrow[x \to 0^+]{} 1$

so $\dfrac{\sin x}{x} \xrightarrow[x \to 0^+]{} 1$. Moral exercise: $x \to 0^-$ works as well.