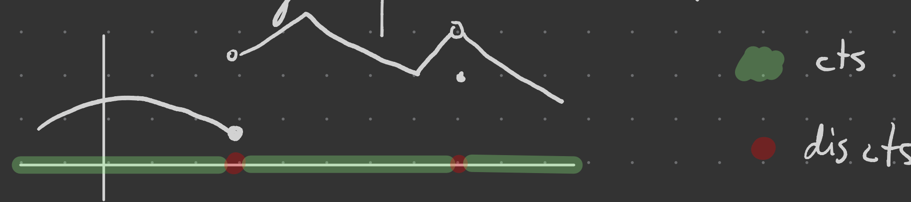
24. IX. 13

Continuity

A function f is continuous at $x = a$ when

$$\lim_{x \to a} f(x) = f(a).$$

A function f is continuous on an open interval when it is continuous at every point in that interval.

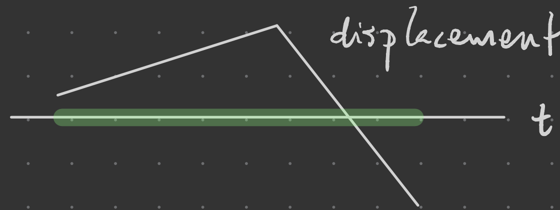


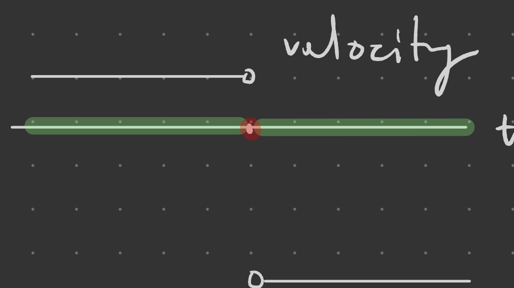
E.g.

- Polynomial functions are continuous on $\mathbb{R}$
- Rational functions are continuous on their domains.
- Trig functions are continuous on their domains.

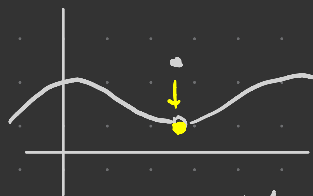
- Composites of cts functions are cts.

E.g. A fly travels east at a constant speed until it impacts a westbound freight train:

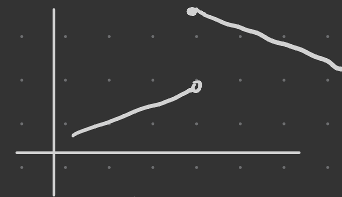




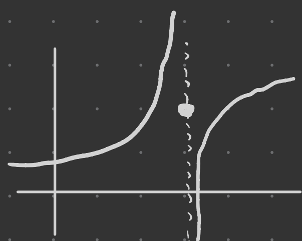
Types of discontinuity



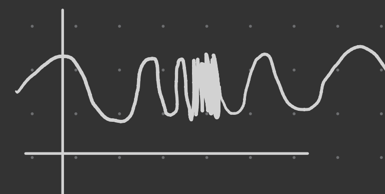
removable



jump

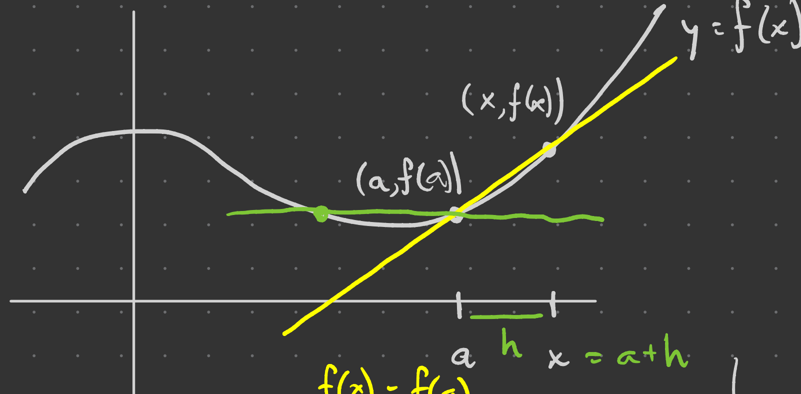


infinite



essential

Secants



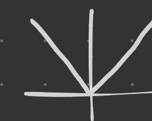
slope of secant

$$m_{sec} = \frac{f(x) - f(a)}{x - a}$$

$$= \frac{f(a+h) - f(a)}{h}$$

difference quotients

- $\lim\limits_{x \to a}$ allows $x < a$
- $\lim\limits_{h \to 0}$ allows $h < 0$



$$\lim_{x \to 0} \frac{|x| - |0|}{x - 0}$$

DNE

Tangents

$x \longrightarrow a$ (or $h \longrightarrow 0$)

$$m_{tan} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

E.g. Let's find the tangent line to $f(x) = x^2$ at $x = 3$:

$$m_{tan} = \lim_{x \to 3} \frac{f(x) - f(3)}{x - 3}$$

$$= \lim_{x \to 3} \frac{x^2 - 9}{x - 3}$$

$$= \lim_{x \to 3} \frac{(x-3)(x+3)}{x-3}$$

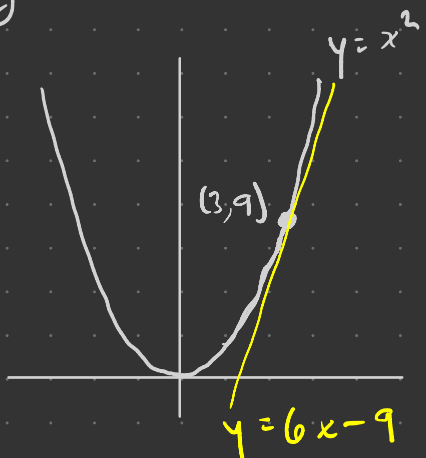
$$= \lim_{x \to 3} (x+3)$$

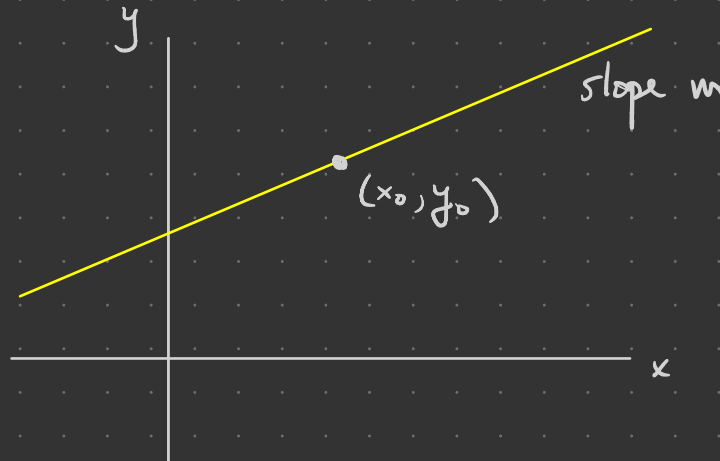
$$= 6$$

The tangent line passes through $(3,9)$ with slope 6, so

$$y - 9 = 6(x-3)$$

$$\Leftrightarrow \quad y = 6x - 9$$





point-slope: line has eq'n

$$y - y_0 = m(x - x_0)$$

$(x_0, y_0) = (3, 9)$ $\quad m = m_{tan} = 6$

Note Also have

$$m_{tan} = \lim_{h \to 0} \frac{f(3+h) - f(3)}{h}$$

$$= \lim_{h \to 0} \frac{(3+h)^2 - 9}{h}$$

$$= \lim_{h \to 0} \frac{9 + 6h + h^2 - 9}{h} = \lim_{h \to 0} \frac{\cancel{h}(6+h)}{\cancel{h}}$$

$$= \lim_{h \to 0} (6+h)$$

$= 6$. as before ✓

Problem Find m_{tan} at $x=4$ for $f(x)=\sqrt{x}$ via one of limit def'ns.

$$m_{tan} = \lim_{x\to 4} \frac{\sqrt{x}-\sqrt{4}}{x-4}$$

$x-4 = (\sqrt{x}-2)\cdot(\sqrt{x}+2)$

$\Diamond$ Is $\sqrt{1}+\sqrt{4}=\sqrt{5}$?

then is

$\sqrt{x}+\sqrt{y}=\sqrt{x+y}$?

NO!

$$= \lim_{x\to 4} \frac{\sqrt{x}-2}{x-4}\cdot\frac{\sqrt{x}+2}{\sqrt{x}+2}$$

$$= \lim_{x\to 4} \frac{\cancel{x-4}}{\cancel{(x-4)}(\sqrt{x}+2)} = \lim_{x\to 4} \frac{1}{\sqrt{x}+2} = \frac{1}{4}$$

Derivatives

Defn Let $f(x)$ be a function defined on an open interval containing a. The derivative of $f(x)$ at a is

$$f'(a) := \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

provided the limit exists.

I.e., $f'(a) = m_{tan}$!

Rate of change

$$m_{sec} = \frac{f(x)-f(a)}{x-a} = \frac{\Delta f}{\Delta x} = \text{average rate of change over } [a, x]$$

Δf: change in f

Δx: change in $x = h$

So $m_{tan} = f'(a) =$ instantaneous rate of change

If $f(t)$ is displacement or position at time t,

then $f'(t)$ is velocity